Problem 1. Find a basis for the null space of the following matrix:

$$\begin{pmatrix} 1 & 1 & 5 & 0 & -3 \\ 1 & 2 & 8 & 0 & -7 \\ -2 & 0 & -4 & 1 & 0 \\ 2 & -1 & 1 & 0 & 6 \end{pmatrix}$$

Solution: We find the reduced row echelon form

$$\begin{pmatrix} 1 & 1 & 5 & 0 & -3 \\ 1 & 2 & 8 & 0 & -7 \\ -2 & 0 & -4 & 1 & 0 \\ 2 & -1 & 1 & 0 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 5 & 0 & -3 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 2 & 6 & 1 & -6 \\ 0 & -3 & -9 & 0 & 12 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We construct a basis vector in the null space from each free variable:

$$\left\{ \begin{pmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} -1 \\ 4 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\}$$

Problem 2. Find a basis for the subspace of $\mathbb{R}^3$ spanned by

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Solution: Construct the matrix with these vectors as its columns and put it into row echelon form:

$$\begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 4 & 4 & 1 & 0 \\ 3 & 6 & 5 & 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 2 & -3 & -2 \\ 0 & 0 & 2 & -3 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 2 & -3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We conclude that first and third columns are pivot columns. Thus, a basis for the column space is

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} \right\}.$$

Problem 3. Compute the change-of-basis matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ where

$$\mathcal{B} = \left\{ \begin{pmatrix} 3 \\ 2 \\ 9 \end{pmatrix}, \begin{pmatrix} 8 \\ 12 \\ 8 \end{pmatrix}, \begin{pmatrix} 10 \\ 2 \\ 6 \end{pmatrix} \right\} \text{ and } \mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

are bases of $\mathbb{R}^3$.

Solution: We put $[P_{\mathcal{C}}\ P_{\mathcal{B}}]$ in reduced row echelon form:

$$\begin{pmatrix} 1 & 0 & 3 & 3 & 8 & 10 \\ 2 & 2 & 0 & 2 & 12 & 2 \\ 3 & 0 & 1 & 9 & 8 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 & 3 & 8 & 10 \\ 0 & 2 & -6 & -4 & -4 & -18 \\ 0 & 0 & -8 & 0 & -16 & -24 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 3 & 3 & 8 & 10 \\ 0 & 1 & -3 & -2 & -2 & -9 \\ 0 & 0 & -1 & 0 & -2 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 3 & 2 & 1 \\ 0 & 1 & 0 & -2 & 4 & 0 \\ 0 & 0 & 1 & 0 & 2 & 3 \end{pmatrix}$$

This new matrix is $\left[I_3\ \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right]$, so we obtain

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{pmatrix} 3 & 2 & 1 \\ -2 & 4 & 0 \\ 0 & 2 & 3 \end{pmatrix}.$$

Determine whether each of the following statements are true or false. No justification is necessary.

Problem 4. A subset H of a vector space V is a subspace of V if and only if the zero vector is in H.

Solution: False. The subset $H = \left\{ \mathbf{0}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$ in $\mathbb{R}^2$ contains $\mathbf{0}$. However, it is not closed under addition, so is not a subspace.

Problem 5. The set of polynomials of degree exactly 2 form a subspace of all polynomials.

Solution: False. $f(x) = x^2 - 2$ and $g(x) = -x^2 + x$ are both in the set, but $(f+g)(x) = x - 2$ is not. Thus it cannot be a vector space.

Problem 6. The column space of an $m \times n$ matrix is $\mathbb{R}^m$.

Solution: False. Consider the zero matrix.

Problem 7. The kernel of a linear transformation is a vector space.

Solution: True. The proof of this is essentially the same as that of Theorem 2 in §4.2.

Problem 8. Let $\mathcal{B}$ be a set of vectors in a vector space V. If $V = \operatorname{span}(\mathcal{B})$ and $\mathcal{B}$ is linearly independent, then $\mathcal{B}$ is a basis for V.

Solution: True. This is the definition of a basis.

Problem 9. Let $S = \{p_1, \ldots, p_n\}$ be a set of polynomials such that no two have the same degree. Prove that S is linearly independent.

Solution: Assume there exists a linear combination

$$c_1 p_1 + \cdots + c_n p_n = 0$$

where not all c_i are zero. Among all indices i such that $c_i \neq 0$ there must exist a unique p_i with largest degree d. For every other index j, either $c_j = 0$ or p_j has smaller degree. Thus, the coefficient x^d in

$$c_1 p_1 + \cdots + c_n p_n$$

must be $c_j \neq 0$. This is a contradiction. Thus S is linearly independent.

Alternative: (This is longer and not as clear, but shows how one could use matrices if one were so inclined.) Let d be the largest degree among the polynomials in S. Let $V = \mathbb{P}_d$ be the space of polynomials of degree at most d and let $\mathcal{B} = \{1, x, \ldots, x^d\}$ be the usual basis for $\mathbb{P}_d$. After possibly reindexing, we may assume that

$$\deg(p_1) < \deg(p_2) < \cdots < \deg(p_n) \ .$$

Consider the $n \times r$ matrix

$$A = ([p_1]_{\mathcal{B}} \cdots [p_n]_{\mathcal{B}}) \,.$$

By the definition of degree, we have $A_{ij} = 0$ whenever $j > \deg(p_i)+1$ and $A_{ij} \neq 0$ whenever $j = \deg(p_i)+1$ Suppose there exists a non-zero vector $\mathbf{v}$ in $\mathrm{Nul}(A)$. Let i be the largest index such that $\mathbf{v}_i \neq 0$. Let $j = \deg(p_i) + 1$. We have

$$(A\mathbf{v})_j = \sum_{k=1}^{d+1} A_{jk}\mathbf{v}_k \ .$$

Since the degrees are strictly increasing, $A_{jk} = 0$ if $j < i$. By construction, $A_{jk} = 0$ if $j > i$. Thus $(A\mathbf{v})_j = A_{ik} \neq 0$. This is a contradiction. Thus $\mathrm{Nul}(A) = \{\mathbf{0}\}$ and we conclude that the columns of A are linearly independent. Thus $\{[p_i]_{\mathcal{B}}\}$ are linearly independent. Thus S is linearly independent.

Comments: You cannot assume that the degrees are already in increasing order, but you can state that they should be reordered to make that happen. You definitely cannot assume that $\deg(p_k) = k$ for each index k. You cannot assume every degree occurs. You cannot assume that p_i is the only polynomial in S with a term of degree $\deg(p_i)$: this is only true for the largest degree polynomial. You cannot assume that non-trivial linear relations have all non-zero weights. Don't conflate polynomials p_i, their indexes i, their degree, and their coordinate vectors; these are all different things.

Problem 10. Suppose that W_1 and W_2 are subspaces of a finite-dimensional vector space V. Prove that

$$\dim(W_1 \cap W_2) \leq \min\left(\dim(W_1), \dim(W_2)\right).$$

Solution: Recall Theorem 12 from §4.5: if H is a subspace of a finite-dimensional vector space U, then H is finite-dimensional and $\dim(H) \leq \dim(U)$. Thus, W_1 and W_2 are finite-dimensional. Now $W_1 \cap W_2$ is a subspace of W_1, so $\dim(W_1 \cap W_2) \leq \dim(W_1)$. Similarly, $\dim(W_1 \cap W_2) \leq \dim(W_2)$. The result follows.

Comments: There are many slight variations of the above that are acceptable proofs. You did not need to cite the specific theorem number above, but you did need to state clearly any result you were appealing to. There were many misconceptions. Many conflated $W_1 \cap W_2$ with a basis for the subspace. In particular, $\dim(W_1 \cap W_2)$ is not the cardinality of the set $W_1 \cap W_2$.
