Consider the matrices

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -2 \\ 2 & 1 \\ 3 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

For each of the following, compute the matrix or indicate that the expression is undefined.

Problem 1. AA^T

Solution:

$$AA^T = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 8 & 13 \end{pmatrix}$$

Problem 2. $C^{-1} + AB$

Solution:

$$C^{-1} + AB = \frac{1}{1 \cdot 4 - 2 \cdot 3} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & 1 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} 9 & -1 \\ 11 & -4 \end{pmatrix} = \begin{pmatrix} 7 & 0 \\ \frac{25}{2} & -\frac{9}{2} \end{pmatrix}$$

Problem 3. Find the inverse of the matrix

$$\begin{pmatrix} 1 & 1 & 2 \\ 3 & -1 & 1 \\ 0 & 2 & 2 \end{pmatrix}$$

or show that it does not exist.

Solution: To invert A, we find the row reduction of $[A\ I_n]$:

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 & 0 & 1 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -4 & -5 & -3 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -1 & -3 & 1 & 2 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 0 & -2 & 1 & \frac{3}{2} \\ 0 & 1 & 0 & -3 & 1 & \frac{5}{2} \\ 0 & 0 & 1 & 3 & -1 & -2 \end{pmatrix}$$

Thus, the inverse is

$$\begin{pmatrix} -2 & 1 & \frac{3}{2} \\ -3 & 1 & \frac{5}{2} \\ 3 & -1 & -2 \end{pmatrix}$$

Problem 4. Find the determinant of the following matrix:

$$\begin{pmatrix} 1 & 0 & 4 & 7 \\ 2 & 1 & 5 & 6 \\ 0 & 0 & 0 & 4 \\ 1 & 8 & 0 & 3 \end{pmatrix}$$

Solution: If the matrix is A, then the cofactor expansion along the third row gives

$$\det(A) = -4 \det \begin{pmatrix} 1 & 0 & 4 \\ 2 & 1 & 5 \\ 1 & 8 & 0 \end{pmatrix}.$$

Now, using the first column we have

$$\det(A) = -4 \left(\det \begin{pmatrix} 1 & 5 \\ 8 & 0 \end{pmatrix} + 4 \begin{pmatrix} 2 & 1 \\ 1 & 8 \end{pmatrix} \right) = -4\,(-40 + 4(15)) = -80.$$

Determine whether each of the following statements are true or false. No justification is necessary.

Problem 5. Let A be an $m \times n$ matrix and let B, C be $n \times p$ matrices. Then $AB + AC = A(B + C)$.

Solution: True. This is Theorem 2.2(b) in the text.

Problem 6. If A is invertible, then the inverse of A^{-1} is A itself.

Solution: True. This is Theorem 2.6(a) in the text.

Problem 7. Let A be an $n \times n$ matrix. If there is an $n \times n$ matrix D such that $AD = I_n$, then there is also an $n \times n$ matrix C such that $CA = I$.

Solution: True. This follows from the equivalence of (j) and (k) in Theorem 2.8 in the text.

Problem 8. The determinant of a triangular matrix is the sum of the entries on the main diagonal.

Solution: False. Consider $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. We have $\det(A) = 1 \neq 1 + 1$.

Problem 9. Let A, B be $n \times n$ matrices. Then $\det(A + B) = \det(A) + \det(B)$.

Solution: False. Consider $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. We have $1 = 0 + 0$; a contradiction.

Problem 10. Let A, B be $n \times n$ matrices. Prove that, if B is invertible, then $\det(A) = \det(B^{-1}AB)$.

Solution: Recall that $\det(CD) = \det(C)\det(D)$ for any two $n \times n$ matrices C, D. Using this, we find

$$\det(B^{-1}AB) = \det(B^{-1})\det(AB) = \det(B^{-1})\det(A)\det(B).$$

Now, since determinants are real numbers, their multiplication is commutative. Thus we may continue:

$$\det(B^{-1})\det(A)\det(B) = \det(A)\det(B)^{-1}\det(B) = \det(A)\det(B^{-1}B) = \det(A)\det(I_n) = \det(A)$$

and obtain the desired equality.

Problem 11. Let A, B be upper triangular $n \times n$ matrices. Prove that AB is also upper triangular.

Solution: Recall that a matrix is upper triangular if and only if every entry below the main diagonal is 0. Thus, a matrix C is upper triangular if and only if, for all indices $1 \leq i, j \leq n$, the condition $i > j$ implies the entry C_{ij} is 0.

Suppose $1 \leq i, j \leq n$ and $i > j$. Consider an integer k where $1 \leq k \leq n$. If $k < i$, then $A_{ik} = 0$ since A is upper triangular. Otherwise, if $k \geq i$, then $k \geq i > j$. Thus $k > j$ and $B_{kj} = 0$ since B is upper triangular. Thus $A_{ik}B_{kj} = 0$ for all k. We conclude

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} = \sum_{k=1}^{n} 0 = 0.$$

Thus AB is also upper triangular.