Problem 1. Find the reduced row echelon form for the matrix

$$\begin{pmatrix} 2 & 1 & 6 & 2 \\ 4 & 1 & 0 & 2 \\ 3 & 1 & 2 & 0 \end{pmatrix}.$$

Solution:

$$\begin{pmatrix} 2 & 1 & 6 & 2 \\ 4 & 1 & 0 & 2 \\ 3 & 1 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & 6 & 2 \\ 0 & -1 & -12 & -2 \\ 3 & 1 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & 6 & 2 \\ 0 & -1 & -12 & -2 \\ 0 & -\frac{1}{2} & -7 & -3 \end{pmatrix}$$
$$\sim \begin{pmatrix} 2 & 1 & 6 & 2 \\ 0 & -1 & -12 & -2 \\ 0 & 0 & -1 & -2 \end{pmatrix} \sim \begin{pmatrix} 2 & 1 & 6 & 2 \\ 0 & 1 & 12 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & -6 & 0 \\ 0 & 1 & 12 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$
$$\sim \begin{pmatrix} 2 & 0 & -6 & 0 \\ 0 & 1 & 0 & -22 \\ 0 & 0 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 0 & 12 \\ 0 & 1 & 0 & -22 \\ 0 & 0 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -22 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

Problem 2. Describe all solutions of $A\mathbf{x} = \mathbf{b}$ in parametric vector form, where

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 4 & 0 & 5 \\ 0 & 0 & 0 & 1 & 6 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}.$$

Solution: Observe that the matrix A is already in reduced row echelon form. If $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$, then the basic variables are x_1, x_2, x_4 and the free variables are x_3, x_5. Thus, the parametric vector form of the solution set is

$$\mathbf{x} = \begin{pmatrix} 7 \\ 8 \\ 0 \\ 9 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ -4 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -3 \\ -5 \\ 0 \\ -6 \\ 1 \end{pmatrix}.$$

Problem 3. Determine if the vectors

$$\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \text{ and } \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

are linearly independent.

Solution: Let A be the matrix with these vectors as columns. The vectors are linearly independent if and only if the homogeneous matrix equation $A\mathbf{x} = \mathbf{0}$ has no non-trivial solutions. By Gaussian elimination, we obtain

$$\begin{pmatrix} 5 & 1 & -1 \\ -2 & 2 & 2 \\ 1 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ -2 & 2 & 2 \\ 5 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 6 & 4 \\ 0 & -9 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 6 & 4 \\ 0 & 0 & 0 \end{pmatrix}.$$

This is in echelon form and not every column has a pivot. Therefore there must be a non-trivial solution. Thus the vectors are linearly dependent.

(Note that $\mathbf{v}_1 - 2\mathbf{v}_2 + 3\mathbf{v}_3 = 0$ is an explicit solution, but we do not need find this to solve the problem.)

Problem 4. Let $A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$ and define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$. Find the image under T of $\mathbf{u} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$.

Solution:

$$T(\mathbf{u}) = A\mathbf{u} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \cdot 5 + 3 \cdot (-2) \\ 4 \cdot 5 + 1 \cdot (-2) \end{pmatrix} = \begin{pmatrix} 4 \\ 18 \end{pmatrix}$$

Problem 5. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear transformation where $T(\mathbf{e}_1) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $T(\mathbf{e}_2) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Find $\mathbf{x}$ such that $T(\mathbf{x}) = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$.

Solution: We need to solve the matrix equation $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$. We form the augmented matrix and use Gaussian elimination:

$$\begin{pmatrix} 3 & 2 & 7 \\ 1 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 3 \\ 3 & 2 & 7 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & -1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

We conclude that $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ works.

Determine whether each of the following statements are true or false. No justification is necessary.

Problem 6. An inconsistent system has more than one solution.

Solution: False. An inconsistent system has no solutions.

Problem 7. The echelon form of a matrix is unique.

Solution: False. Consider $\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, which are row equivalent and both in echelon form. (The reduced echelon form is unique.)

Problem 8. If A is an $m \times n$ matrix whose columns do not span $\mathbb{R}^m$, then the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b}$ in $\mathbb{R}^n$.

Solution: True. If the columns of A do not span $\mathbb{R}^m$, then there exists $\mathbf{b} \in \mathbb{R}^m$ that is not a linear combination of the columns of A. Thus $A\mathbf{x} = \mathbf{b}$ has no solutions.

Problem 9. If $\mathbf{x}$ is a nontrivial solution of $A\mathbf{x} = \mathbf{0}$, then every entry in $\mathbf{x}$ is nonzero.

Solution: False. For the matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, the vector $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a nontrivial solution with a zero entry.

Problem 10. Any set containing the zero vector is linearly dependent.

Solution: True. Given $\mathbf{0}, \mathbf{v}_1, \dots, \mathbf{v}_n$, for some non-negative n, we always have the nontrivial relation

$$1\mathbf{0} + 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_n.$$

Problem 11. Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span $\mathbb{R}^n$. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation such that $T(\mathbf{v}_i) = 0$ for all i from $1, \ldots, p$. Prove that $T(\mathbf{x}) = 0$ for every vector $\mathbf{x} \in \mathbb{R}^n$.

Solution: Let $\mathbf{x} \in \mathbb{R}^n$ be arbitrary. Since $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span $\mathbb{R}^n$, there must exist weights $c_1, \ldots, c_p$ in $\mathbb{R}$ such that

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p.$$

Now by the superposition principle, we have

$$\begin{aligned} T(\mathbf{x}) &= T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) \\ &= c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p) \\ &= \mathbf{0} + \cdots + \mathbf{0} \\ &= \mathbf{0} \end{aligned}$$

as desired.