University of South Carolina

MATH 544-001

Final Examination Practice A

April 30, 2024

Closed book examination   Time: 150 minutes

Instructions:

Notes, books, computer, phones, calculators or other aids are **not** allowed. Please write on only one side of each page. If you need more space than is provided, then ask for extra paper from the proctor. Simplify your final answers. Full credit will not be awarded for insufficient accompanying work.

There are 60 total points available.

Problem 1. (5 points) Find the reduced row echelon form for the matrix

$$\begin{pmatrix} 1 & 5 & 2 & 6 \\ 2 & 5 & 1 & 2 \\ 2 & 1 & 0 & 3 \end{pmatrix}.$$

Problem 2. (5 points) Describe all solutions of the following linear system.

$$\begin{aligned} x_1 + x_2 + x_3 + 3x_4 &= 0 \\ 2x_1 + x_2 + 3x_3 + 4x_4 &= 1 \\ x_2 + x_4 &= 3 \end{aligned}$$

Problem 3. (5 points) Consider the matrices

$$A = \begin{pmatrix} 4 & 6 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 3 \\ 2 & 3 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

For each of the following, compute the matrix or indicate that the expression is undefined.

AC

BC

$B - AC$

C^{-1}

Problem 4. (5 points) Find the inverse of the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

or show that it does not exist.

Problem 5. (5 points) Find the determinant of the following matrix:

$$\begin{pmatrix} 3 & 1 & 1 \\ 4 & 5 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Problem 6. (5 points) The set

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 4 \\ 5 \end{pmatrix} \right\}$$

is a basis for $\mathbb{R}^4$. Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ where

$$\mathbf{x} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 1 \end{pmatrix}.$$

Problem 7. (5 points) If it exists, find an invertible matrix P such that $A = PDP^{-1}$ where

$$A = \begin{pmatrix} 0 & 2 & 2 \\ 1 & 2 & 1 \\ 4 & -2 & 2 \end{pmatrix} \text{ and } D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

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Determine whether each of the following statements are true or false. No justification is necessary.

Problem 8. (1 point) A system of linear equations never has exactly 2 distinct solutions.

Problem 9. (1 point) A mapping $T : \mathbf{R}^n \to \mathbf{R}^m$ is surjective if each vector in $\mathbf{R}^n$ maps onto a unique vector in $\mathbf{R}^m$.

Problem 10. (1 point) If S is a linearly independent set, then no vector in S is a linear combination of the other vectors in S.

Problem 11. (1 point) Suppose A, B are matrices. If $AB = BA$, then A and B are square matrices.

Problem 12. (1 point) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation with standard matrix A, then $\det(T) = \det(A)$.

Determine whether each of the following statements are true or false. No justification is necessary.

Problem 13. (1 point) Let A, B be $n \times n$ matrices. Then $(AB)^T = A^T B^T$.

Problem 14. (1 point) Every vector space has a finite basis.

Problem 15. (1 point) Let $T : V \to V$ be a linear transformation from a vector space V to itself. If $\mathcal{B}$ and $\mathcal{C}$ are finite bases for V, then $[T]_{\mathcal{B}}$ is similar to $[T]_{\mathcal{C}}$

Problem 16. (1 point) Every square matrix is similar to a diagonal matrix.

Problem 17. (1 point) Suppose A is a square matrix with an eigenvalue λ of algebraic multiplicity m. If E_λ is the eigenspace associated to λ, then $\dim(E_\lambda) > m$.

Problem 18. (5 points) Let $\mathbb{P}$ be the vector space of all polynomials in one variable x. Let V be the subset of polynomials $f \in \mathbb{P}$ such that $f(2) = 0$. Prove that V is a subspace of V.

Problem 19. (5 points) A square matrix A is orthogonal if $A^T A = I_n$. Prove that a square matrix A is orthogonal if and only if $AA^T = I_n$.

Problem 20. (5 points) Let A and B be similar matrices. Prove that the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is injective if and only if the linear transformation $\mathbf{x} \mapsto B\mathbf{x}$ is injective.

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