Problem A. Let A and B be $n \times n$ matrices. Prove that if A is similar to B and A is invertible, then B is invertible and A^{-1} is similar to B^{-1}.

Solution. Recall that A is similar to B if there exists an invertible matrix P such that $A = PBP^{-1}$. First, observe that $B = P^{-1}AP$. Consider the matrix $C = P^{-1}A^{-1}P$. Observe that

$$BC = (P^{-1}AP)(P^{-1}A^{-1}P) = P^{-1}AA^{-1}P = P{-}1P = I_n.$$

By part (k) of the invertible matrix theorem, B is invertible with inverse $B^{-1} = C$.

Problem B. Let M_n be the set of $n \times n$ matrices. Define a relation $\sim$ on M_n where $A \sim B$ means A is similar to B. Prove that $\sim$ is an equivalence relation.

Solution. We need to show that $\sim$ is reflexive, symmetric, and transitive.

Reflexive: If $A \in M_n$, then $A = I_n A I_n^{-1}$. Thus $A \sim A$.

Symmetric: Suppose $A \sim B$. Then $A = PBP^{-1}$ for an invertible matrix P. Rearranging, we see that $B = P^{-1}AP$. Taking $Q = P^{-1}$, we see that $B = QAQ^{-1}$ for an invertible matrix Q. Thus $B \sim A$.

Transitive: Suppose $A \sim B$ and $B \sim C$. Then $A = PBP^{-1}$ for an invertible matrix P and $B = QCQ^{-1}$ for an invertible matrix Q. Let $R = PQ$. We see that

$$A = PBP^{-1} = P(QCQ^{-1})P^{-1} = (PQ)C(PQ)^{-1} = RCR^{-1}$$

and conclude that $A \sim C$.