Problem A. Let $T : V \to W$ be an invertible linear transformation from a vector space V to a vector space W. Prove that if $v_1, \ldots, v_p$ is a basis for V, then $T(v_1), \ldots, T(v_p)$ is a basis for W.

Solution. From Assignment 7B, we conclude that $T(v_1), \ldots, T(v_p)$ is linearly independent. From Assignment 7C, we conclude that $T(v_1), \ldots, T(v_p)$ spans W. Thus, it is a basis as desired.

Problem B. Let $\mathcal{B}$ be a finite basis for a vector space V. Prove that the vectors $u_1, \ldots, u_p$ in V are linearly independent if and only if the coordinate vectors $[u_1]_{\mathcal{B}}, \ldots, [u_p]_{\mathcal{B}}$ are linearly independent.

Solution. Recall that the map $T : V \to \mathbb{R}^n$ given by $T(x) = [x]_{\mathcal{B}}$ is an invertible linear transformation. Thus, this follows immediately from Problem A.

Problem C. Let U, V, W be vector spaces and let $T : U \to V$ and $S : V \to W$ be linear transformations. Prove that if $\text{im}(T) \cap \ker(S) = \{0\}$ then $\ker(T) = \ker(S \circ T)$.

Solution. Suppose $x \in \ker(T)$. T hen $S(T(x)) = S(0) = 0$. Thus $x \in \ker(S \circ T)$.

Now suppose $x \in \ker(S \circ T)$. Then $S(T(x)) = 0$. Then $T(x) \in \ker(S)$. Since also $T(x) \in \text{im}(T)$, we conclude that $T(x) \in \{0\}$. Thus $T(x) = 0$. Thus $x \in \ker(T)$.

Problem D. Let $a_1, \ldots, a_n$ be distinct real numbers. Recall from Assignment 7 that, for each $1 \leq i \leq n$, there is a unique polynomial f_i of degree $n - 1$ such that $f_i(a_j) = \delta_{ij}$ for all $1 \leq j \leq n$. Prove that $\mathcal{B} = \{f_1, \ldots, f_n\}$ is a basis for the vector space $\mathbb{P}_{n-1}$ of polynomials of degree $\leq n - 1$.

Solution. First, we show that $\mathcal{B}$ is linearly independent. Suppose there exists a relation

$$0 = \sum_{i=1}^{n} c_i f_i$$

for some $c_1, \ldots, c_n \in \mathbb{R}$. Then, whenever $1 \leq j \leq n$, we obtain

$$0 = \left(\sum_{i=1}^{n} c_i f_i \right) (a_j) = \sum_{i=1}^{n} c_i f_i(a_j) = c_j.$$

Thus $c_j = 0$ for all $1 \leq j \leq n$. Thus $\mathcal{B}$ is linearly independent.

Since $\{1, x, \ldots, x^{n-1}\}$ is a basis $\mathbb{P}_{n-1}$, we conclude that $\mathbb{P}_{n-1}$ has dimension n. Since $\mathcal{B}$ is a linearly independent set of size n, it must be a basis.