Problem A. Let $T : V \to W$ be a linear transformation from a vector space V to a vector space W. Prove that the range of T is a subspace of W.

Solution. Let U be the range of T. We check the properties given in the definition of being a subspace. First, we observe that $T(0) = 0$ so $0 \in W$. Second, if $u, v \in U$, then there exist $x, y \in V$ such that $u = T(x)$ and $v = T(y)$. Thus $u + v = T(x) + T(y) = T(x + y)$ is in the range of T. Finally, if $c \in \mathbb{R}$, then $cu = cT(x) = T(cx)$ is in the range of T. We conclude that U is a subspace.

Problem B. Let $T : V \to W$ be a linear transformation from a vector space V to a vector space W. Suppose $v_1, \dots, v_p$ is a linearly independent set in V. Prove that, if T is injective, then $T(v_1), \dots, T(v_p)$ is linearly independent.

Solution. We prove that contrapositive. Suppose $T(v_1), \dots, T(v_p)$ is linearly dependent. Then there exist weights $c_1, \dots, c_p$, not all zero, such that

$$c_1T(v_1) + \cdots + c_pT(v_p) = 0 \ .$$

By linearity, this means

$$T(c_1v_1 + \cdots + c_pv_p) = 0 \ .$$

Since T is injective, this means

$$c_1v_1 + \cdots + c_pv_p = 0 \ .$$

This means $v_1, \dots, v_p$ is linearly dependent.

Problem C. Let $T : V \to W$ be a linear transformation from a vector space V to a vector space W. Suppose $v_1, \dots, v_p$ spans V. Prove that, if T is surjective, then $T(v_1), \dots, T(v_p)$ spans W.

Solution. Consider some $w \in W$. Since T is surjective, there exists a $v \in V$ such that $w = T(v)$. Since $v_1, \dots, v_p$ span V, there exist weights $c_1, \dots, c_p$ such that

$$v = \sum_{i=1}^{p} c_iv_i \ .$$

Thus

$$w = T(v) = T(\sum_{i=1}^{p} c_iv_i) = \sum_{i=1}^{p} c_iT(v_i).$$

We conclude that w is a linear combination of $T(v_1), \dots, T(v_p)$. Since w was arbitrary, they form a spanning set for W as desired.

Problem D. Let $a_1, \ldots, a_n$ be distinct real numbers. For each $1 \le i \le n$, prove that there is a unique polynomial f_i of degree $n-1$ such that $f_i(a_j) = \delta_{ij}$ for all $1 \le j \le n$. (Here δ_{ij} is the Kronecker delta.)

Solution. Fix $1 \le i \le n$. We observe that

$$f_i(x) = \prod_{j=1, i \neq j}^{n} \frac{x - a_i}{a_j - a_i}$$

has the desired properties. Indeed, $f_i(a_i) = 1$ and $f_i(a_j) = 0$ if $j \neq i$. Moreover, it is a product of $n-1$ linear factors and therefore has degree $n-1$. Thus the polynomial f_i exists.

Let us prove that f_i is unique. Suppose f_i, g are polynomials of degree $n-1$ such that $f_i(a_j) = g(a_i)$ for all $1 \le j \le n$. Let $h = f_i - g$. Observe that $h(a_j) = 0$ for all $1 \le j \le n$. If $h(a_j) = 0$, then $(x - a_j)$ divides $h(x)$. Thus

$$h(x) = j(x) \prod_{i=1}^{n} (x - a_i)$$

for some polynomial $j(x)$. If $j \neq 0$, then $\deg(h) = \deg(j) + n \ge n$; this contradicst that the degree of h is $n-1$. Thus we conclude that $j = 0$; thus $h = 0$; thus $g = f_i$. We conclude f_i is unique.