

**Problem A.** A matrix  $A$  is symmetric if  $A = A^T$ . Prove that the set of symmetric matrices form a subspace of the vector space of all  $n \times n$  matrices.

Solution. Let  $V$  be the vector space of all  $n \times n$  matrices and let  $S$  be the subset of symmetric matrices. We verify the three properties in the definition of a subspace. First,  $0^T = 0$  so the zero matrix is symmetric. Second, if  $A, B$  are symmetric matrices, then  $(A+B)^T = A^T + B^T = A+B$ ; thus  $A+B$  is also symmetric. Finally, if  $c$  is a scalar and  $A$  is a symmetric matrix, then  $(cA)^T = cA^T = cA$ ; thus  $cA$  is symmetric.

**Problem B.** Let  $U, V$  be subspaces of a vector space  $W$ . Prove that  $U \cap V$  is a subspace of  $W$ . Find an example where  $U \cup V$  is not a subspace of  $W$ .

Solution. We verify the three defining properties of a vector space. Since  $0 \in U$  and  $0 \in V$ , we conclude  $0 \in U \cap V$ . If  $u, v \in U \cap V$ , then  $u, v \in U$  and  $u, v \in V$ . Since  $U$  is a vector space,  $u+v \in U$ . Since  $V$  is a vector space,  $u+v \in V$ . Thus  $u+v \in U \cap V$ . If  $u \in U \cap V$  and  $c \in \mathbb{R}$ , then  $cu \in U$  and  $cu \in V$ . Thus  $cu \in U \cap V$ .

For the second statement, observe that

$$U = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \mid x \in \mathbb{R} \right\}, \quad W = \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix} \mid y \in \mathbb{R} \right\}$$

are subspaces of  $\mathbb{R}^2$ . Consider:

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u+v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

in  $\mathbb{R}^2$ . Observe that  $u \in U \subseteq U \cup V$  and  $v \in V \subseteq U \cup V$ , but  $u+v$  is in neither  $U$  nor  $V$ . Thus  $U \cup V$  is not a subspace.