Problem A. Let A be an invertible matrix. Prove that $\det(A^{-1}) = \det(A)^{-1}$.

Solution.

We have $A^{-1}A = I_n$. Thus $\det(A^{-1}A) = \det(I_n)$. Note that $\det(I_n) = 1$. By the multiplicativity of the determinant, we have $\det(A^{-1})\det(A) = 1$. Solving, we obtain $\det(A^{-1}) = \det(A)^{-1}$ as desired.

Problem B. Let A be an $n \times n$ matrix and let r be a scalar. Prove that $\det(rA) = r^n \det(A)$.

Solution.

Observe that $rA = rI_nA$. Since rI_n is a diagonal matrix with n entries of r along the diagonal, we have $\det(rI_n) = r^n$. Now $\det(rA) = \det(rI_n)\det(A) = r^n \det(A)$ as desired.

Alternatively: We define a series of matrices inductively as follows. Let $B_0 = A$. For $1 \leq i \leq n$, let B_i be the matrix B_{i-1} with the ith row multiplied by r. Thus, $rA = B_n$. By Theorem 3.3(c),

$$\det(rA) = \det(B_n) = r\det(B_{n-1}) = r^2\det(B_{n-2}) = \cdots = r^n\det(B_0) = r^n\det(A).$$