**Problem A.** Let $m$ and $b$ be real numbers. Let $f : \mathbb{R}^1 \to \mathbb{R}^1$ be the function given by $f(x) = mx + b$. Prove that $f$ is a linear transformation if and only if $b = 0$.

Solution.

Suppose $f$ is a linear transformation. Then $f(2) = 2f(1)$. Since $f(2) = 2m + b$ and $f(1) = m + b$, we have $2m + b = 2(m + b)$, which simplifies to $b = 0$ as desired.

Now suppose that $b = 0$. For any $u, v \in \mathbb{R}^1$, we have

$$f(u + v) = m(u + v) = mu + mv = f(u) = f(v).$$

For any $u \in \mathbb{R}^1$ and $c \in \mathbb{R}$, we have

$$f(cu) = m(cu) = c(mu) = cf(u).$$

Thus, by the definition, $f$ is a linear transformation.

**Problem B.** Let $A$ be a fixed $n \times n$ matrix. Prove that $AB = BA$ for all $n \times n$ matrices $B$ if and only if $A = \lambda I_n$ for some real number $\lambda$.

Solution.

Suppose $A = \lambda I_n$ for some real number $\lambda$ and $B$ is an $n \times n$ matrix. Then

$$AB = (\lambda I_n)B = \lambda(I_n B) = \lambda B = \lambda(BI_n) = B(\lambda I_n) = BA.$$

Now, suppose $AB = BA$ for all $n \times n$ matrices $B$. The result holds for $n = 1$ since multiplication of real numbers satisfies $ab = ba$; thus we may suppose $n \geq 2$. Suppose $1 \leq i, j \leq n$ and $i \neq j$.

Let $C(i, j)$ be the matrix with a 1 in the $i, j$ entry and a 0 in every other entry. Observe that

$$[AC(i,j)]_{ij} = \sum_{k=1}^{n} A_{ik} C(i,j)_{kj} = A_{ii},$$

while

$$[C(i,j)A]_{ij} = \sum_{k=1}^{n} C(i,j)_{ik} A_{kj} = A_{jj}.$$

Similarly,

$$[AC(i,j)]_{jj} = \sum_{k=1}^{n} A_{jk} C(i,j)_{kj} = A_{ji},$$

while

$$[C(i,j)A]_{jj} = \sum_{k=1}^{n} C(i,j)_{jk} A_{kj} = 0$$

since $j \neq i$. Since $AC(i,j) = C(i,j)A$, by considering the $ij$-entry of this matrix equation, we conclude that $A_{ii} = A_{jj}$. By considering the $jj$-entry, we conclude that $A_{ji} = 0$. Since $i, j$ were any two non-equal indices, we conclude that $A_{ij} = 0$ whenever $i \neq j$ and

$$A_{11} = A_{22} = \cdots = A_{nn}.$$

Setting $\lambda = A_{11}$, we have $A = \lambda I_n$ as desired.

This "Σ notation" proof might seem mysterious. Things may be clearer with a specific $3 \times 3$ example:

$$AC(1,2) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_{11} & 0 \\ 0 & a_{21} & 0 \\ 0 & a_{31} & 0 \end{pmatrix}$$

and

$$C(1,2)A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{22} & a_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$