**Problem A.** Let $X$ and $Y$ be finite subsets of $\mathbb{R}^n$. Prove that $\text{Span}(X \cap Y) \subseteq \text{Span}(X) \cap \text{Span}(Y)$.

Solution.

Suppose $\mathbf{v}$ is an element of $\text{Span}(X \cap Y)$. By definition, this means that $\mathbf{v}$ is a linear combination of elements of $X \cap Y$. This means that

$$\mathbf{v} = \sum_{\mathbf{w} \in X \cap Y} c_{\mathbf{w}} \mathbf{w}$$

where each $c_{\mathbf{w}}$ is a real number. Observe that

$$\mathbf{v} = \sum_{\mathbf{w} \in X \cap Y} c_{\mathbf{w}} \mathbf{w} + \sum_{\mathbf{w} \in X \setminus Y} 0\mathbf{w},$$

so $\mathbf{v}$ is a linear combination of the elements in $X$. Thus $\mathbf{v} \in \text{Span}(X)$. Similarly,

$$\mathbf{v} = \sum_{\mathbf{w} \in X \cap Y} c_{\mathbf{w}} \mathbf{w} + \sum_{\mathbf{w} \in Y \setminus X} 0\mathbf{w},$$

so $\mathbf{v} \in \text{Span}(Y)$. We conclude that $\mathbf{v} \in \text{Span}(X) \cap \text{Span}(Y)$.

**Problem B.** Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be column vectors in $\mathbb{R}^n$. Prove that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent if and only if $\{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{u}+\mathbf{w}\}$ is linearly independent.

Solution.

Assume $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent. Consider real numbers $a, b, c$ such that

$$a(\mathbf{u}+\mathbf{v}) + b(\mathbf{v}+\mathbf{w}) + c(\mathbf{u}+\mathbf{w}) = 0.$$

Rearranging, we obtain

$$(a+c)\mathbf{u} + (a+b)\mathbf{v} + (b+c)\mathbf{w} = 0.$$

Since $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ are linearly independent, this implies

$$a + c = a + b = b + c = 0.$$

This is a linear system with unique solution $a = b = c = 0$, which can be computed using Gaussian elimination. Since the trivial solution is the unique solution, we conclude $\{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{u}+\mathbf{w}\}$ is linearly independent.

Conversely, suppose $\{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{u}+\mathbf{w}\}$ is linearly independent. Consider real numbers $a, b, c$ such that

$$a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = 0.$$

By rearranging, we obtain

$$\frac{a+b-c}{2}(\mathbf{u}+\mathbf{v}) + \frac{-a+b+c}{2}(\mathbf{v}+\mathbf{w}) + \frac{a-b+c}{2}(\mathbf{u}+\mathbf{w}) = 0.$$

By linear independence, we see that

$$\frac{a+b-c}{2} = \frac{-a+b+c}{2} = \frac{a-b+c}{2} = 0.$$

By Gaussian elimination, this linear system has the unique solution $a = b = c = 0$. Since the trivial solution is the unique solution, we conclude $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent.