Problem A. Let A be an $n \times n$ matrix. Prove that if λ is an eigenvalue of A, then λ is an eigenvalue of A^T.

Solution. If λ is an eigenvalue of A, then $B = A - \lambda I_n$ is not invertible. Thus B^T is not invertible. Now

$$B^T = (A - \lambda I_n)^T = A^T - \lambda I_n^T = A^T - \lambda I_n.$$

Thus $A^T - \lambda I_n$ is not invertible. Thus A^T has λ as an eigenvalue.

Problem B. Let A be an invertible $n \times n$ matrix. Prove that if A is diagonalizable, then A^{-1} is diagonalizable.

Solution. By definition, A is similar to a diagonal matrix D. Thus, $A = PDP^{-1}$ for an invertible matrix P. Observe that $D = P^{-1}AP$. Since a product of invertible matrices is invertible, we conclude that D is invertible. Thus

$$A^{-1} = \left(PDP^{-1}\right)^{-1} = (P^{-1})^{-1}D^{-1}P^{-1} = PD^{-1}P^{-1}.$$

The inverse of a diagonal matrix is also diagonal. Thus A^{-1} is similar to a diagonal matrix.