Symmetric Groups and General Linear Groups

Alexander Duncan

March 10, 2023

Here we discuss the theory of symmetric functions, with the particular goal of describing representations of the symmetric groups and general linear groups. The irreducible representations of the symmetric group S_n are the *Specht modules* V_λ, which are parametrized by the partitions λ of weight n. The irreducible polynomial representations of the general linear group $\mathrm{GL}(\mathbb{V})$ are precisely the (images of the) *Schur functors* $\mathbb{S}_\lambda(V)$ where λ is partition of length $\ell(\lambda) \leq \dim(V)$. Both are best understood as being in bijection with *Schur functions* $\{s_\lambda\}$, which form an orthonormal basis for the ring of symmetric functions.

These notes are not self-contained. Many proofs will be sketched or left as a reference. This is not because the proofs are hard (the beauty of this subject is that they are often very slick!), but that the theory is too rich to properly explore in only a few weeks. Much of this material is drawn from [FH91, §4,6,A], [EGH+11, §5.12–5.19], [Mac95, §I], and [Sta99, §7].

1 Partitions

Recall that a *partition* λ is a sequence of non-strictly decreasing non-negative integers $\lambda_1 \geq \lambda_2 \geq \cdots$ that is eventually 0. Some standard terminology:

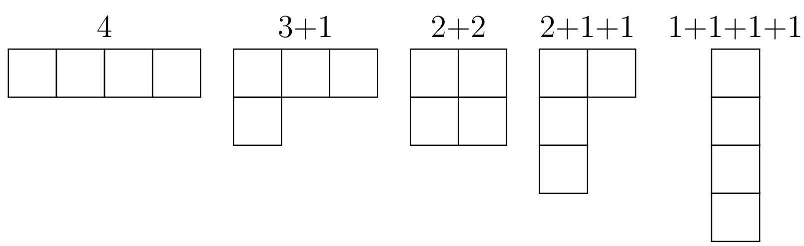
- The non-zero λ_i are the *parts* of λ.
- The number $\ell(\lambda)$ of parts is the *length* of λ.
- The sum $|\lambda| = \sum_{i \geq 0} \lambda_i$ is the *weight* of λ.
- A "partition of n" is a partition of weight n.

- $\lambda \vdash n$ means λ is a partition of n.
- The number $m_i(\lambda)$ of parts equal to i is the *multiplicity* of i in λ.
- Partitions can be written $\lambda_1 + \lambda_2 + \cdots + \lambda_r$.
- We may use the shorthand $\lambda = (1^{m_1} 2^{m_2} \cdots r^{m_r})$.
- $\lambda + \mu$ is the partition $(\lambda_1 + \mu_1, \ldots)$.
- $\lambda \subseteq \mu$ means $\lambda_i \leq \mu_i$ for all $i \geq 1$.
- The dominance ordering $\lambda \leq \mu$ means $\sum_{j=1}^{i} \lambda_j \leq \sum_{j=1}^{i} \mu_j$ for all $i \geq 1$.

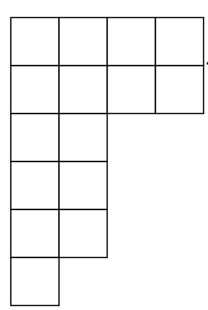
Partitions are often drawn as *Young diagrams*. This is just a series of empty boxes where each row contains λ_i boxes. We use *English notation* where λ_1 is the top row. There are some differing conventions between different areas of math, so read any source carefully.

Given a partition λ, the *conjugate partition* is the partition $\lambda^\dagger$ obtained by reflecting the Young diagram in the downwards-right diagonal line. More explicitly, $\lambda_i^\dagger$ is the number of parts such that $\lambda_j \geq i$.

Example 1.1. The partitions of 4 are as follows:



Example 1.2. Let $\lambda = (4, 4, 2, 2, 2, 1)$. We may also write λ as $4 + 4 + 2 + 2 + 2 + 1$ or $1^1 2^3 4^2$. We have parts $\lambda_1 = 4$, $\lambda_2 = 4$, $\lambda_3 = 2$, $\lambda_4 = 2$, $\lambda_5 = 2$, and $\lambda_6 = 1$. We have length $\ell(\lambda) = 6$, weight $|\lambda| = 15$, and multiplicities $m_1(\lambda) = 1$, $m_2(\lambda) = 3$, $m_3(\lambda) = 0$, and $m_4(\lambda) = 2$. The Young tableau is



.

The conjugate partition $\lambda^\dagger$ is $(6, 5, 2, 2)$.

Recall that every permutation $\sigma \in S_n$ can be written as a product of disjoint cycles, which is unique up to reordering of the cycles. Counting the cycles of length 1, we see that the orders of the constituent cycles give a partition $\lambda \vdash n$. For example $(1\ 4\ 3)(2\ 8)(6\ 7) \in S_9$ has cycle type $3 + 2 + 2 + 1 + 1$. The following is standard in most group theory texts:

Proposition 1.3. *Two permutations in S_n are conjugate if and only if they have the same cycle type. In particular, the conjugacy classes of S_n are in canonical bijective correspondence with partitions of n.*

For every partition λ, we define the integers

$$\epsilon_\lambda = (-1)^{|\lambda|-\ell(\lambda)}.$$

and

$$z_\lambda = \prod_{i\geq 1} i^{m_i} m_i!$$

where $m_i = m_i(\lambda)$ denotes the multiplicities,

Exercise 1.4. *Prove that, if $\sigma \in S_n$ has cycle type λ, then* $\mathrm{sgn}(\sigma) = \epsilon_\lambda$.

Exercise 1.5. *Prove that, if $\sigma \in S_n$ has cycle type λ, then the centralizer $Z_{S_n}(\sigma)$ has order z_λ. Equivalently, the number of elements of S_n with cycle type λ is $n!/z_\lambda$.*

2 Symmetric Polynomials

There is a natural left action of the symmetric group S_n on the ring of polynomials $R = \mathbb{Z}[x_1, \ldots, x_n]$ by permuting the variables. More precisely, there is a unique ring automorphism of R defined by $x_i \mapsto x_{\sigma(i)}$ for each variable x_i and each permutation $\sigma \in S_n$. Alternatively, if $\sigma \in S_n$, the we have

$$(\sigma \cdot f)(a_1, \ldots, a_n) = f\left(a_{\sigma(1)}, \ldots, a_\sigma(n)\right)$$

for $f \in R$ and $a_1, \ldots, a_n \in \mathbb{Z}$.

Definition 2.1. A polynomial $f \in \mathbb{Z}[x_1, \ldots, x_n]$ is *symmetric* if $\sigma \cdot f = f$ for all $\sigma \in \Sigma$. We denote by SF_n the set of symmetric polynomials $\mathbb{Z}[x_1, \ldots, x_n]^{S_n}$ in n variables.

The set SF_n forms a graded ring

$$\mathsf{SF}_n = \bigoplus_{d \geq 0} \mathsf{SF}_n^d$$

where each $\mathsf{SF}_n^d = \mathbb{Z}[x_1, \ldots, x_n]_{(d)}^{S_n}$ is the subgroup of homogeneous symmetric polynomials of degree d.

Given a tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ we use the shorthand

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

to denote monomials in $\mathbb{Z}[x_1, \ldots, x_n]$.

Given a partition $\lambda \vdash d$ with length $\ell(\lambda) \leq n$, the *monomial symmetric polynomial associated to* λ is given by

$$m_\lambda := \sum_\alpha x^\alpha$$

where $(\alpha_1, \ldots, \alpha_n)$ ranges over all *distinct* permutations of $(\lambda_1, \ldots, \lambda_n)$.

Exercise 2.2. *Prove that* $z_\lambda m_\lambda = \sum_{\sigma \in S_n} x^{\sigma(\lambda)}$ *for all* λ *of length* $\ell(\lambda) \leq n$.

Example 2.3. If $n = 3$, then we have

$$\begin{aligned}
m_\emptyset &= 1 \\
m_1 &= x_1 + x_2 + x_3 \\
m_2 &= x_1^2 + x_2^2 + x_3^2 \\
m_{11} &= x_1x_2 + x_1x_3 + x_2x_3 \\
m_{111} &= x_1x_2x_3 \\
m_{14} &= x_1x_2^4 + x_1^4x_2 + x_1x_3^4 + x_1^4x_3 + x_2x_3^4 + x_2^4x_3.
\end{aligned}$$

Importantly, we have the following:

Proposition 2.4. *Monomial symmetric polynomials form a basis for* SF_n. *Specifically,*

$$\mathsf{SF}_n^d = \mathrm{span}_\mathbb{Z}\{m_\lambda \mid \lambda \vdash d, \ell(\lambda) \leq n\}.$$

Proof. The partitions of length at most n are a system of distinct representatives for the S_n-orbits of $\mathbb{N}^n$. The monomials in m_λ are precisely those of the orbit of λ. The coefficient of every monomial in m_λ is either 0 or 1 and every possible monomial occurs in exactly one m_λ. If f is a symmetric polynomial, then the coefficient of a monomial x^α must be the same as $x^{\sigma(\alpha)}$ for every $\sigma \in S_n$. □

There are several notable additional families of symmetric polynomials which we define now.

Definition 2.5. For positive integers d, we define the *elementary symmetric polynomial* e_d *of degree* d as

$$e_d = m_{1^d} = \sum_{1 \le i_1 < i_2 < \cdots < i_d \le n} x_{i_1} x_{i_2} \cdots x_{i_d},$$

the *complete homogeneous symmetric polynomial* h_d *of degree* d as

$$h_d = \sum_{\lambda \vdash d} m_\lambda = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_d \le n} x_{i_1} x_{i_2} \cdots x_{i_d},$$

and the *power sum symmetric polynomial* p_d *of degree* d as

$$p_d = m_d = \sum_{1 \le i \le n} x_i^d.$$

By convention, $e_0 = h_0 = p_0 = 1$.

Remark 2.6. The conventions for e_0 and h_0 are uncontroversial, but many references leave p_0 undefined. Indeed, there are good reasons to instead define $p_0 = n$, but this does not extend to the ring of symmetric functions discussed below.

Example 2.7. If $n = 3$, then we have

$$\begin{aligned}
e_1 = h_1 = p_1 &= x_1 + x_2 + x_3 \\
e_2 &= x_1x_2 + x_1x_3 + x_2x_3 \\
e_3 &= x_1x_2x_3 \\
e_4 &= 0 \\
h_2 &= x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3 \\
h_3 &= x_1^3 + \cdots + x_1^2x_2 + \cdots + x_1x_2x_3 \\
p_2 &= x_1^2 + x_2^2 + x_3^2 \\
p_3 &= x_1^3 + x_2^3 + x_3^3
\end{aligned}$$

It is worth mentioning right away (but whose proof we will defer) some fundamental results. First, we have *Newton's identities*:

$$de_d = \sum_{i=1}^{d} (-1)^{i-1} p_i e_{d-i},$$

and the fundamental relation

$$0 = \sum_{i=0}^{d} (-1)^i e_i h_{d-i},$$

which hold for all positive integers d. These allow one to recursively compute p_i's and h_i's in terms of e_i's (and vice versa).

We also have the *Fundamental Theorem of Symmetric Polynomials* which states that $e_1, \ldots, e_n$ are algebraically independent generators of the ring SF_n. In fact, combining this with the above relations, we have

$$\begin{aligned}
\mathbb{Z}[x_1, \ldots, x_n]^{S_n} &= \mathbb{Z}[e_1, \ldots, e_n] \\
\mathbb{Z}[x_1, \ldots, x_n]^{S_n} &= \mathbb{Z}[h_1, \ldots, h_n] \\
\mathbb{Q}[x_1, \ldots, x_n]^{S_n} &= \mathbb{Q}[p_1, \ldots, p_n]
\end{aligned}$$

where rational coefficients are necessary for the last equality. The proofs of these equalities are not very hard, but we defer them for the moment as they are consequences of more general facts.

2.1 Alternating and Schur Polynomials

Let $\epsilon : S_n \to \{\pm 1\}$ be the sign homomorphism and let $A_n = \ker(\epsilon)$ be the alternating group on n letters.

Definition 2.8. A polynomial $f \in \mathbb{Z}[x_1, \ldots, x_n]$ is *alternating* if $\sigma \cdot f = \epsilon(\sigma) f$ for all $\sigma \in S_n$. Given an element $\alpha \in \mathbb{N}^n$, define the *alternant of* α as

$$a_\alpha = \sum_{\sigma \in S_n} \epsilon(\sigma) x^{\sigma(\alpha)},$$

which is an alternating polynomial.

Note that $a_\alpha = 0$ if and only if the entries of α are not distinct. We also see that every alternating polynomial is a linear combination of a_λ's for partitions λ where the parts λ_i are all distinct. Observe that all parts of λ are distinct if and only if $\delta \subseteq \lambda$ where $\delta = (n-1, n-2, \ldots, 1, 0)$. We have the following:

Lemma 2.9. *The set*

$$\{a_\lambda \mid \ell(\lambda) \leq n, \delta \subseteq \lambda\}$$

is a basis for the group of alternating polynomials.

Example 2.10. The *Vandermonde polynomial* $\Delta \in \mathbb{Z}[x_1, \ldots, x_n]$ is defined via

$$\Delta = \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

Note that $(i\ j) \cdot \Delta = -\Delta$ for all $i \neq j$. We conclude that $\sigma(\Delta) = \mathrm{sgn}(\sigma)$ for all $\sigma \in S_n$. Thus Δ is alternating. Observe that $\Delta = a_\delta$.

Lemma 2.11. *Every alternating polynomial f has the form $g\Delta$ where g is a symmetric polynomial.*

Proof. It suffices to show that $(x_i - x_j)$ divides f for all $i \neq j$. Indeed, suppose $c_\alpha x^\alpha$ is a monomial in f and write $x^\alpha = x_i^u x_j^v x^\beta$ where x_i and x_j do not divide x^β. Since f is alternating, $-c_\alpha x_i^v x_j^u x^\beta$ also must appear in f. Thus $c_\alpha(x_i^u x_j^v - x_i^v x_j^u)x^\beta$ occurs in f. Each of these divisible by $(x_i - x_j)$ as desired. □

The following definition now makes sense:

Definition 2.12. For a partition λ of length $\leq n$, the *Schur polynomial* is the symmetric polynomial $s_\lambda = a_{\lambda+\delta}/a_\delta$.

The alternants $\{a_{\lambda+\delta}\}$ are a basis for the alternating polynomials analogously to the monomial symmetric polynomials $\{m_\lambda\}$ being a basis for the symmetric polynomials. Since we have $a_{\lambda+\delta} = s_\lambda a_\delta$, the Schur polynomials as "monomial alternating polynomials" except we've divided out the alternating part so that they are symmetric.

In particular, we have the following:

Proposition 2.13. *The set of Schur polynomials*

$$\{s_\lambda \mid \lambda \vdash d, \ell(\lambda) \leq n\}$$

are a basis for SF_n^d.

The *Kostka numbers* $K_{\lambda\mu}$ are the entries of the change of basis matrix between the monomial basis and the Schur basis. Specifically, they are non-negative integers such that

$$s_\lambda = \sum_{\substack{\mu \vdash |\lambda| \\ \ell(\mu) \leq n}} K_{\lambda\mu} m_\mu$$

for all partitions λ, μ. The Kostka numbers do not depend on the number of variables n in the ambient polynomial ring (we do not prove this here).

Example 2.14. For $n \geq 4$, we have the following change of basis matrix:

$$\begin{pmatrix} s_{1111} \\ s_{211} \\ s_{22} \\ s_{31} \\ s_{4} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} m_{1111} \\ m_{211} \\ m_{22} \\ m_{31} \\ m_{4} \end{pmatrix}$$

Thus, for example $K_{(31)(1111)} = 3$.

Exercise 2.15. *Prove that* $s_{(d)} = h_d$ *and* $s_{(1^d)} = e_d$.

We will not need the following later, but it is of general interest:

Theorem 2.16 (Fundamental Theorem of Alternating Polynomials)**.** *The ring* $\mathbb{Z}[x_1, \ldots, x_n]^{A_n}$ *is a free* SF_n*-module with basis* $\{1, \Delta\}$*. In other words, every* A_n*-invariant polynomial* f *can be written uniquely as* $f = p + \Delta q$ *where* p, q *are symmetric.*

Remark 2.17. If V is an n-dimensional complex linear representation of G, then we have a natural action of G on the polynomial ring $S = \mathbb{C}[x_1, \ldots, x_n]$ by viewing it as the symmetric algebra $\mathcal{S}(V^\vee)$. A famous theorem of Hochster-Roberts proves that the invariant ring $S^G = \mathbb{C}[x_1, \ldots, x_n]^G$ is a Cohen-Macaulay ring: there exists a polynomial subring $R = \mathbb{C}[f_1, \ldots, f_n]$ such that S^G is a free R-module. The "Fundamental Theorem of Alternating Polynomials" can be seen as a very special case of this fact (where the subring and basis have particular interpretations).

2.2 Symmetric and Exterior Powers

We now see some of the connections of symmetric functions and representation theory:

Proposition 2.18. *Suppose* V *is an* n*-dimensional vector space and* $\varphi \in \mathrm{End}(V)$ *has eigenvalues* $\alpha_1, \ldots, \alpha_n$*. Then the natural action of* φ *on the exterior power* $\Lambda^d V$ *has trace*

$$\mathrm{tr}\left(\Lambda^d \varphi\right) = e_d(\alpha_1, \ldots, \alpha_n)$$

and the natural action of φ *on the symmetric power* $\mathcal{S}^d V$ *has trace*

$$\mathrm{tr}\left(\mathcal{S}^d \varphi\right) = h_d(\alpha_1, \ldots, \alpha_n).$$

Proof. It suffices to work over an algebraically closed field k. We prove the statement for $\Lambda^d V$ since the argument for $\mathcal{S}^d$ is very similar.

First, we assume that φ is diagonalizable with eigenvectors $v_1, \ldots, v_n$ corresponding to eigenvalues $\alpha_1, \ldots, \alpha_n$. Now, an eigenbasis for $\Lambda^d V$ is given by

$$\{v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_d} \mid 1 \leq i_1 < i_2 < \cdots < i_d \leq n\} .$$

The corresponding eigenvalues of $\Lambda^d \varphi$ are therefore

$$\{\alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_d} \mid 1 \leq i_1 < i_2 < \cdots < i_d \leq n\} .$$

The trace of $\Lambda^d \phi$ is just the sum of the eigenvalues, which is $e_d(\alpha_1, \ldots, \alpha_n)$ as desired.

Now we consider the case where φ is not diagonalizable. In this case φ has a Jordan canonical form. Thus, we have a basis $v_1, \ldots, v_n$ for V such that φ is represented by $D + N$ where D is a diagonal matrix and N is a lower triangular matrix. Observe that $N(v_i)$ is a linear combination of $v_{i+1}, \ldots, v_n$.

Define a total ordering on $\mathbb{N}^n$ where $(a_1, \ldots, a_n) < (b_1, \ldots, b_n)$ if $a_i < b_i$ for the minimal i on which $a_i \neq b_i$. We now observe that

$$(D + N)(v_{i_1} \wedge \cdots \wedge v_{i_d}) = D(v_{i_1} \wedge \cdots \wedge v_{i_d}) + \text{higher terms}$$

where the "higher terms" are scalar multiples of $v_{j_1} \wedge \cdots \wedge v_{j_d}$ where we have $(j_1, \ldots, j_d) > (i_1, \ldots, i_d)$. Thus, appropriately ordered, our basis for $\Lambda^d V$ also represents $\Lambda^d V$ as a lower-triangular matrix. The trace only depends on the diagonal entries so the result follows from the diagonalizable case. $\square$

Another way of understanding the previous proposition is that $\Lambda^d V$ and $\mathcal{S}^d V$ are representations of $\mathrm{GL}(V)$ with corresponding characters e_d and h_d, respectively. We will see that there is a class of symmetric polynomials called *Schur polynomials* which are precisely the characters of the irreducible *polynomial* representations of $\mathrm{GL}(V)$.

3 The Ring of Symmetric Functions

Many of the relations between various special symmetric polynomials are basically independent of the number n of variables $x_1, \ldots, x_n$ in the ambient polynomial ring. The *ring of symmetric functions* is a standard object in algebraic combinatorics that facilitates this. Several equivalent constructions exist, but we will follow the construction from [Sta99], which is fairly down to earth.

Let $\mathbb{Z}[[\{x_n\}_{n\in\mathbb{N}_{>0}}]]$ be the ring of formal power series in countably many variables. Recall that this means that we allow only finitely many x_i in each monomial, but each element may be a linear combination of infinitely many monomials and there may be no upper bound on the subscripts i occurring among the x_i's in each monomial. For a non-negative integer d, the subgroup $\mathbb{Z}[[\{x_n\}_{n\in\mathbb{N}_{>0}}]]_{(d)}$ consists of those elements whose monomials all have degree exactly d (in other words, exactly d variables x_i in each monomial, counting multiplicities).

Let $S_{\mathbb{N}_{>0}}$ be the symmetric group on $\mathbb{N}_{>0}$; in other words, the group of all bijections $\mathbb{N}_{>0} \to \mathbb{N}_{>0}$. There is a natural action of $S_{\mathbb{N}_{>0}}$ on $\mathbb{Z}[[\{x_n\}_{n\in\mathbb{N}_{>0}}]]$ by permuting variables that preserves the degree. A *homogeneous symmetric function of degree* d is an element $f \in \mathbb{Z}[[\{x_n\}_{n\in\mathbb{N}_{>0}}]]_{(d)}^{S_{\mathbb{N}}}$. Let SF^d be the subgroup of homogeneous symmetric functions of degree d.

We define the *ring of symmetric functions* as the (internal) direct sum

$$\mathsf{SF} = \bigoplus_{d\geq 0} \mathsf{SF}^d,$$

which is a graded subring of $\mathbb{Z}[[\{x_n\}_{n\in\mathbb{N}_{>0}}]]$.

Remark 3.1. Note that "symmetric function" is standard terminology, but it's not a great name since it's not really clear what it is a function *of*. Moreover, the term symmetric function is often used in the finite variable case to discuss "ordinary" functions like e^{x+y}, which are invariant under symmetries like $x \leftrightarrow y$.

There are canonical surjective graded ring homomorphisms $\rho_n : \mathsf{SF} \to \mathsf{SF}_n$, which are defined via

$$\rho_n(f)(x_1, \ldots, x_n) = f(x_1, \ldots, x_n, 0, 0, \cdots).$$

Note that convergence is not an issue since all but finitely many monomials will evaluate as zero.

Remark 3.2. One can equivalently define the ring of symmetric functions as an inverse limit

$$\mathsf{SF} = \varprojlim_n \mathsf{SF}^n$$

in the category of graded rings (see [Mac95]). (Warning: it is *not* the inverse limit in ordinary rings.) The ρ_n above are precisely the canonical projections obtained from this construction.

We define the *monomial symmetric function associated to* λ given by

$$m_\lambda := \sum_\alpha x^\alpha$$

where $(\alpha_1, \ldots) \in \mathbb{N}^{\mathbb{N}_{>0}}$ ranges over all *distinct* permutations of $(\lambda_1, \ldots)$. Observe that the set $\{m_\lambda\}$ is a basis for SF.

We can now define the *elementary symmetric functions*

$$e_d = m_{1^d},$$

the *complete homogeneous symmetric functions*

$$h_d = \sum_{\lambda \vdash d} m_\lambda,$$

and the *power sum symmetric functions*

$$p_d = m_d.$$

The overloading of m_λ, e_d, h_d, and p_d to mean different objects in each SF^n is seen to be mostly harmless in view of the fact that they are precisely the images of the corresponding objects in SF under the maps to ρ_n.

We may also define the *Schur symmetric functions* s_λ via

$$s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$$

by taking advantage of the fact that the Kostka numbers do not depend on the number of variables in the given polynomial ring.

3.1 Relations and Identities

Proposition 3.3. $\displaystyle\sum_{i=0}^{d} (-1)^i e_i h_{d-i} = 0$ *for every* $d \geq 1$.

Proof. Consider the generating functions

$$E(t) = \sum_{r \geq 0} e_r t^r \text{ and } H(t) = \sum_{r \geq 0} h_r t^r$$

as elements in $\mathsf{SF}[[t]]$. We see that

$$\begin{aligned}&E(t)\\=&1+(x_1+x_2+\cdots)t+(x_1x_2+x_1x_3+\cdots)t^2\\=&\prod_{i\geq 1}(1+x_it)\end{aligned}$$

in the larger ring $\mathbb{Z}[[x_1,\ldots]][[t]]$. Similarly, recalling the geometric series formula, we have

$$H(t)=\prod_{i\geq 1}(1-x_it)^{-1}.$$

The equality $H(t)E(-t)=1$ is immediate. The desired identities are simply the coefficients of t^d in this equality. □

Proposition 3.4 (Newton Identities)**.** *For all $d\geq 1$,*

$$dh_d=\sum_{i=1}^{d}p_ih_{d-i} \text{ and}$$
$$de_d=\sum_{i=1}^{d}(-1)^{i-1}p_ie_{d-i}.$$

Proof. We compute

$$\begin{aligned}\sum_{r\geq 1}p_rt^{r-1}&=\sum_{i\geq 1}\sum_{d\geq 0}x_i^dt^{d-1}=\sum_{i\geq 1}\frac{x_i}{1-x_it}\\&=\sum_{i\geq 1}\frac{d}{dt}\log\left((1-x_i)^{-1}\right)=\frac{H'(t)}{H(t)}=\frac{E'(-t)}{E(-t)}\end{aligned}$$

and then we read the identities off from the coefficients of t^d in $P(t)H(t)=H'(t)$ and $P(t)E(-t)=E'(-t)$. □

For $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_r)$ we define $e_\lambda=e_{\lambda_1}\cdot e_{\lambda_2}\cdots e_{\lambda_r}$, $h_\lambda=h_{\lambda_1}\cdot h_{\lambda_2}\cdots h_{\lambda_r}$, and $p_\lambda=p_{\lambda_1}\cdot p_{\lambda_2}\cdots p_{\lambda_r}$.

Recall the *dominance partial ordering* on partitions where $\lambda\geq\mu$ if and only if $\lambda_1+\cdots+\lambda_i\geq\mu_1+\cdots\mu_i$ for all $i\geq 1$. (Note this is only a partial ordering as, for example, (31^3) and (2^3) are incomparable.

Proposition 3.5. *We have*

$$e_\lambda = \sum_{\mu \vdash |\lambda|} M_{\lambda\mu} m_\mu$$

where $M_{\lambda\mu}$ is the number of $\{0,1\}$-matrices whose rows sum to $\lambda_1, \lambda_2, \ldots$ and whose columns sum to $\mu_1, \mu_2, \ldots$. Moreover, $M_{\lambda\mu} \neq 0$ if and only if $\mu \leq \lambda^\dagger$ and $M_{\lambda\lambda^\dagger} = 1$.

Proof. We merely sketch the argument. We consider the terms of e_λ and m_μ in reference to the following matrix:

$$X = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots \\ x_1 & x_2 & x_3 & \cdots \\ x_1 & x_2 & x_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Observe that each $\{0,1\}$-matrix Y produces a monomial x^α by taking the product of the entries of X corresponding to the non-zero entries of Y. Monomials in e_λ are uniquely constructed by taking the product of exactly λ_1 distinct entries from row 1, exactly λ_2 from row 2, etc. Monomials x^μ are constructed by taking the product of exactly μ_1 distinct entries from column 1, exactly μ_2 from column 2, etc.

The conditions on $M_{\lambda\mu}$ now follow by looking for $\{0,1\}$-matrices satisfying certain constraints. □

After refining our partial order to a total order on the partitions, we can think of $M = (M_{\lambda\mu})$ as an infinite integer matrix. Note that $M_{\lambda\mu^\dagger}$ is triangular with 1s along the diagonal. Thus M is invertible and we have the following important corollary:

Corollary 3.6 (Fundamental Theorem of Symmetric Functions)**.** *There is an equality of rings*

$$\mathsf{SF} = \mathbb{Z}[e_1, e_2, \ldots]$$

where $e_1, e_2, \ldots$ are algebraically independent.

Exercise 3.7. *Find an analog for Proposition 3.5 for h_λ using $\mathbb{N}$-matrices instead of $\{0,1\}$-matrices. Conclude that $\mathsf{SF} = \mathbb{Z}[h_1, h_2, \ldots]$ where $h_1, h_2, \ldots$ are algebraically independent.*

Exercise 3.8. *Use the Newton identites to show that* $\mathsf{SF} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}[p_1, p_2, \ldots]$ *where* $p_1, p_2, \ldots$ *are algebraically independent.*

Example 3.9.

$$\begin{pmatrix} e_{11} \\ e_2 \\ e_{111} \\ e_{21} \\ e_3 \\ e_{1111} \\ e_{211} \\ e_{22} \\ e_{31} \\ e_4 \end{pmatrix} = \begin{pmatrix} 2 & 1 & & & & & & & & \\ 1 & & & & & & & & & \\ & & 6 & 3 & 1 & & & & & \\ & & 3 & 1 & & & & & & \\ & & 1 & & & & & & & \\ & & & & & 24 & 12 & 6 & 4 & 1 \\ & & & & & 12 & 5 & 2 & 1 & \\ & & & & & 6 & 2 & 1 & & \\ & & & & & 4 & 1 & & & \\ & & & & & 1 & & & & \end{pmatrix} \begin{pmatrix} m_{11} \\ m_2 \\ m_{111} \\ m_{21} \\ m_3 \\ m_{1111} \\ m_{211} \\ m_{22} \\ m_{31} \\ m_4 \end{pmatrix}$$

Example 3.10.

$$\begin{pmatrix} h_{11} \\ h_2 \\ h_{111} \\ h_{21} \\ h_3 \\ h_{1111} \\ h_{211} \\ h_{22} \\ h_{31} \\ h_4 \end{pmatrix} = \begin{pmatrix} 2 & 1 & & & & & & & & \\ 1 & 1 & & & & & & & & \\ & & 6 & 3 & 1 & & & & & \\ & & 3 & 2 & 1 & & & & & \\ & & 1 & 1 & 1 & & & & & \\ & & & & & 24 & 12 & 6 & 4 & 1 \\ & & & & & 12 & 7 & 4 & 3 & 1 \\ & & & & & 6 & 4 & 3 & 2 & 1 \\ & & & & & 4 & 3 & 2 & 2 & 1 \\ & & & & & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} m_{11} \\ m_2 \\ m_{111} \\ m_{21} \\ m_3 \\ m_{1111} \\ m_{211} \\ m_{22} \\ m_{31} \\ m_4 \end{pmatrix}$$

Example 3.11.

$$\begin{pmatrix} e_{11} \\ e_2 \\ e_{111} \\ e_{21} \\ e_3 \\ e_{1111} \\ e_{211} \\ e_{22} \\ e_{31} \\ e_4 \end{pmatrix} = \begin{pmatrix} 1 & & & & & & & & & \\ \frac{1}{2} & -\frac{1}{2} & & & & & & & & \\ & & 1 & & & & & & & \\ & & \frac{1}{2} & -\frac{1}{2} & & & & & & \\ & & \frac{1}{6} & -\frac{1}{2} & \frac{1}{3} & & & & & \\ & & & & & 1 & & & & \\ & & & & & \frac{1}{2} & -\frac{1}{2} & & & \\ & & & & & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & & \\ & & & & & \frac{1}{6} & -\frac{1}{2} & & \frac{1}{3} & \\ & & & & & \frac{1}{24} & -\frac{1}{4} & \frac{1}{8} & \frac{1}{3} & -\frac{1}{4} \end{pmatrix} \begin{pmatrix} p_{11} \\ p_2 \\ p_{111} \\ p_{21} \\ p_3 \\ p_{1111} \\ p_{211} \\ p_{22} \\ p_{31} \\ p_4 \end{pmatrix}$$

3.2 Young Tableaux

Now we point out a combinatorial interpretation for the Schur functions and the Kostka numbers. A *semistandard Young tableau T of shape λ* is Young diagram of the partition λ whose boxes are filled with positive integers that are non-strictly increasing in each row and strictly increasing in each column. The *type* or *content* α of a semistandard Young tableau T is the sequence $(\alpha_1, \alpha_2, \ldots)$ of non-negative integers where α_i is the number of boxes containing i. Given a semistandard Young tableau T of type α, we have a monomial

$$x^T = x_1^{\alpha_1} x_2^{\alpha_2} \cdots .$$

Example 3.12. The following is a Young tableau of shape 422 and type $(2, 3, 0, 1, 2)$:

1	1	2	5
2	2		
4	5		

The corresponding monomial is $x^T = x_1^2 x_2^3 x_4 x_5^2$.

The following is used as the *definition* of Schur functions in [Sta99]; that it agrees with the classical definition above is [Sta99, Theorem 7.15.2].

Theorem 3.13. *If λ is a partition, then*

$$s_\lambda = \sum_T x^T$$

where the sum is over all semistandard Young tableaux of shape λ. In particular, $K_{\lambda\mu}$ is the number of semistandard Young tableaux of shape λ and type μ.

Various properties of Kostka numbers are easier to check given this definition:

Exercise 3.14. *For partitions λ, μ, we have $K_{\lambda\mu} = 0$ unless $\mu \leq \lambda$ in the dominance order. Moreover $K_{\lambda\lambda} = 1$.*

In fact, almost all transition matrices of SF can be understood in terms of Kostka numbers (see [Mac95, §I.6] for a comprehensive treatment). Some notable examples:

$$s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$$
$$h_\lambda = \sum_\mu K_{\mu\lambda} s_\mu$$
$$e_\lambda = \sum_\mu K_{\mu^\dagger\lambda} s_\mu.$$

For a partition $\lambda \vdash n$, the specific Kostka number $f^\lambda = K_{\lambda(1)^n}$ is of special interest. Namely, f^λ counts the number of *standard Young tableau* whose boxes contain each of the integers $\{1, \ldots, n\}$ exactly once (while still increasing down rows and columns). For the (i, j)th box of a Young diagram, let the *hook length* $h(i, j)$ count the number of boxes directly below and those directly to the right of the box (i, j) as well as the box itself.

Theorem 3.15 (Hook Length Formula)**.** $f^\lambda = \frac{n!}{\prod h(i,j)}$ *where the product is over all boxes of the young diagram of* λ*.*

Example 3.16. Consider the partition $\lambda = (4, 3, 2)$. We fill in each box of the Young diagram with its hook length:

6	5	3	1
4	3	1	
2	1		

Thus,

$$f^\lambda = \frac{9!}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 3 \cdot 2} = 168$$

in this case.

The change of basis between power sums and Schur functions are especially interesting for the representation theory of symmetric groups as we will see shortly.

Example 3.17.

$$\begin{pmatrix} p_{11} \\ p_2 \\ p_{111} \\ p_{21} \\ p_3 \\ p_{1111} \\ p_{211} \\ p_{22} \\ p_{31} \\ p_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & & & & & & & & \\ -1 & 1 & & & & & & & & \\ & & 1 & 2 & 1 & & & & & \\ & & -1 & 0 & 1 & & & & & \\ & & 1 & -1 & 1 & & & & & \\ & & & & & 1 & 3 & 2 & 3 & 1 \\ & & & & & -1 & -1 & 0 & 1 & 1 \\ & & & & & 1 & -1 & 2 & -1 & 1 \\ & & & & & 1 & 0 & -1 & 0 & 1 \\ & & & & & -1 & 1 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} s_{11} \\ s_2 \\ s_{111} \\ s_{21} \\ s_3 \\ s_{1111} \\ s_{211} \\ s_{22} \\ s_{31} \\ s_4 \end{pmatrix}$$

We point out some other useful constructions (without proof) that hopefully will encourage the reader to delve more deeply into [Mac95] and [Sta99].

The *Hall inner product* is the unique symmetric positive definite bilinear form $(-,-)$ on SF satisfying

$$\begin{aligned} (s_\lambda, s_\mu) &= \delta_{\lambda\mu}, \\ (h_\lambda, m_\mu) &= \delta_{\lambda\mu}, \text{ and} \\ (p_\lambda, p_\mu) &= z_\lambda \delta_{\lambda\mu} \end{aligned}$$

for all partitions λ, μ.

A consequence of the Fundamental Theorem is that there is a ring isomorphism

$$\omega : \mathsf{SF} \to \mathsf{SF}$$

given by $\omega(e_\lambda) = h_\lambda$ for every partition λ. In view of Proposition 3.3, we see that $\omega(h_\lambda) = e_\lambda$ as well. With a bit more work, one can show that $\omega(p_\lambda) = \epsilon_\lambda p_\lambda$ and $\omega(s_\lambda) = s_{\lambda^\dagger}$. This last observation show us that ω is an isometry of the Hall inner product: $(\omega(f), \omega(g)) = (f, g)$ for all $f, g \in \mathsf{SF}$.

4 Representations of Symmetric Groups

Let S_n be the symmetric group on n letters. Recall that the conjugacy classes of S_n are in bijection with partitions of n by taking their cycle types. Given a partition $\lambda \vdash n$ and a class function $f \in C(S_n)$, we define $f(\lambda) := f(\sigma)$

where $\sigma \in S_n$ is any permutation with cycle type λ. Let $\chi_\lambda \in C(S_n)$ denote the characteristic function

$$\chi_\lambda(\sigma) := \begin{cases} 1 & \text{if } \sigma \text{ has cycle type } \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.1. *There is a isometric group isomorphism*

$$\mathrm{ch} : R(S_n) \to \mathsf{SF}^n,$$

called the characteristic map, *defined by*

$$\mathrm{ch}(f) := \sum_{\lambda \vdash n} \frac{f(\lambda)}{z_\lambda} p_\lambda.$$

Viewed as a class function, the values of $f \in R(S_n)$ can be computed via

$$f(\lambda) = (\mathrm{ch}(f), p_\lambda).$$

The characteristic functions $\chi_\lambda \in C(S_n)$ correspond to $\frac{p_\lambda}{z_\lambda}$ in $\mathsf{SF}^n_{\mathbb{Q}}$. The irreducible characters in $R(S_n)$ are exactly the $\chi^\lambda := \mathrm{chi}(s_\lambda)$ for $\lambda \vdash n$. The trivial character $\chi^{(n)}$ corresponds to $h_n = s_n$, and the sign character $\chi^{(1^n)}$ corresponds to $e_n = s_{(1)^n}$. The involution $\omega : \mathsf{SF}^n \to \mathsf{SF}^n$ corresponds to tensoring with the sign character.

Proof. We only point a few nice facts that are easy to see. First we define the *characteristic map* $\mathrm{ch} : C(S_n) \to \mathsf{SF}^n \otimes_{\mathbb{Z}} \mathbb{C}$ since it's not clear that the image of $R(S_n)$ is even contained in SF^n. Recalling that $(p_\lambda, p_\mu) = z_\lambda \delta_{\mu\lambda}$, the formula

$$f(\lambda) = (\mathrm{ch}(f), p_\lambda)$$

is immediate. This shows that ch is bijective (on complex vector spaces).

The correspondence between χ_λ and $\frac{p_\lambda}{z_\lambda}$ follow now from the observation that

$$\chi_\lambda(\mu) = \delta_{\lambda\mu} = \left(\frac{p_\lambda}{z_\lambda}, p_\mu\right)$$

for all λ and μ.

Now, recall that the number of elements in S_n with cycle type λ is given by $\frac{n!}{z_\lambda}$. Thus

$$(\chi_\lambda, \chi_\mu) = \begin{cases} \frac{1}{z_\lambda} & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

But this is exactly the same as (p_λ, p_μ), so ch is an isometry.

We omit the proof of the remaining statements (see [Mac95, §I.7] or [Sta99, §7.18]). □

An immediate corollary of the above theorem is that character table of S_n is precisely the same as the change of basis matrix between the $\{p_\lambda\}$ and $\{s_\lambda\}$ in SF_n. If the entries of the character table of S_n are denoted χ^λ_μ, then

$$\chi_\lambda = \sum_\mu \chi^\mu_\lambda \chi^\mu$$

is equivalent to

$$p_\lambda = \sum_\mu \chi^\mu_\lambda s_\mu.$$

One can now see that the entries of the matrix from Example 3.17 are exactly the entries of the character tables for S_2, S_3, and S_4.

Definition 4.2. The irreducible representation V_λ of S_n corresponding to χ^λ is the *Specht module* associated to λ.

Proposition 4.3. $\dim_{\mathbb{C}} V_\lambda = f^\lambda = \dfrac{n!}{\prod h(i,j)}$

Proof. If $f \in R(S_n)$ is the character of V_λ, then evaluating at the identity gives

$$f(e) = f(1^d) = (\mathrm{ch}(f), p_{(1^d)}) = (s_\lambda, h_{(1^d)}) = K_{\lambda(1)^f} = f^\lambda.$$

Now we appeal to Theorem 3.15. □

Defining the graded group

$$R(S_*) := \bigoplus_{n\geq 0} R(S_n)$$

we obtain a characteristic map

$$\mathrm{ch} : R(S_*) \to \mathsf{SF}$$

by adding together graded components. However, the existing multiplication on each $R(S_n)$ *does not* correspond to the multiplication on SF. We define a multiplication on $R(S_*)$ that makes ch into a graded ring isomorphism.

There is a natural embedding $S_n \times S_m \hookrightarrow S_{n+m}$; there are many ways of doing this, but they are all conjugate. Given a representation ρ of S_n, we obtain a representation $\widetilde{\rho}$ of $S_n \times S_m$ from ρ by precomposition with the first projection. Similarly, given a representation σ of S_m, we obtain a representation $\widetilde{\sigma}$ of $S_n \times S_m$.

Let $R(S_n)$ be the representation ring of the symmetric group S_n. We define a bilinear multiplication

$$\boxtimes : R(S_n) \times R(S_m) \to R(S_{n+m})$$

via

$$\rho \boxtimes \sigma := \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} (\widetilde{\rho} \otimes \widetilde{\sigma}).$$

This turns $R(S_*)$ into a commutative, graded ring and ch is a graded ring isomorphism.

4.1 Explicit Description of Specht modules

Specht modules can also be described without reference to symmetric functions (see [FH91, §4] and [EGH$^+$11, §5.11-17]).

Here we actually construct each Specht module V_λ as a subrepresentation of the regular representation of S_n. Note that the isotypic component associated to V_λ usually has multiplicity greater than 1, so the construction unsurprisingly relies on an arbitrary choice.

Definition 4.4. A *Young tableau* T is a Young diagram for a partition λ of n where each number in $\{1, \dots, n\}$ is assigned to exactly one box. (Note that this neither a special case nor a generalization of the standard and semistandard Young tableau discussed above.) Given a Young tableau T, we define P_T as the subgroup of S_n that permutes only the numbers within each row of T and Q_T as the subgroup that permutes only the numbers within each column of T.

Example 4.5. The following is a Young tableau for the partition 322:

3	1	5
6	2	
4	7	

We have

$$P_T = S_{\{1,3,5\}} \times S_{\{2,6\}} \times S_{\{4,7\}} \text{ and } Q_T = S_{\{3,4,6\}} \times S_{\{1,2,7\}}$$

where S_X is the group of permutations of the set X.

If $\lambda = (\lambda_1, \dots, \lambda_r)$ is a partition, define $S_\lambda := S_{\lambda_1} \times \cdots \times S_{\lambda_r}$. Observe that $P_T \cong S_\lambda$ and $Q_T \cong S_{\lambda^\dagger}$. The subgroups P_T and Q_T depend on the choice of tableau T, but the conjugacy class only depends on the corresponding partition λ.

Given a Young tableau T and a representation W of S_n, define the *Young projectors*:

$$a_T := \frac{1}{|P_T|} \sum_{\sigma \in P_T} \sigma \text{ and } b_T := \frac{1}{|Q_T|} \sum_{\sigma \in Q_T} \epsilon(\sigma)\sigma$$

which are easily seen to be projections in $\mathrm{End}(W)$. The *Young symmetrizer* is the endomorphism $c_T = a_T \circ b_T$. One checks that $c_T = b_T \circ a_T$.

Theorem 4.6. *Suppose λ is a partition of n. If V_{reg} is the regular representation and T is a Young tableau for λ, then the image of $c_T(V_{reg})$ is isomorphic to the Specht module V_λ.*

Proof. We only sketch this. First, let $U_T = a_T(V_{reg})$ and $W_T = b_T(V_{reg})$. We observe that

$$U_T \cong \mathrm{Ind}_{P_T}^{S_n} 1_{P_T}$$

and

$$W_T \cong \mathrm{Ind}_{Q_T}^{S_n} \epsilon_{Q_T}$$

where 1_{P_T} is the trivial representation and ϵ_{Q_T} is the sign representation. In other words, the characteristic of U_T is

$$\mathrm{ch}(1 \boxtimes \cdots \boxtimes 1) = h_{\lambda_1} \cdots h_{\lambda_r} = h_\lambda,$$

while the characteristic of W_T is

$$\mathrm{ch}(\epsilon \boxtimes \cdots \boxtimes \epsilon) = e_{\lambda_1^\dagger} \cdots e_{\lambda_r^\dagger} = e_{\lambda^\dagger}.$$

Writing out h_λ and $e_{\lambda^\dagger}$ in the Schur basis, we see that s_λ is the only basis vector with a non-zero entry in both. Moreover, $(s_\lambda, h_\lambda) = (s_\lambda, e_{\lambda^\dagger}) = 1$. Thus, the image $c_T = a_T \circ b_T$ is isomorphic to the Specht module V_λ provided the image is non-zero.

Let $\{v_\sigma\}_{\sigma\in S_n}$ be the standard basis for the regular representation V_{reg}. We have the concrete description

$$c_T(v_1) = \frac{1}{|P_T||Q_T|}\sum_{\sigma\in P_T}\sum_{\tau\in Q_T}\epsilon(\tau)v_{\sigma\tau}.$$

Since $P_T \cap Q_T = \{1\}$, the $v_{\sigma\tau}$ are all distinct. Thus $c_T \neq 0$ as desired. □

5 Representations of $\mathrm{GL}(V)$

Schur functors are described directly in [FH91, §6] and [EGH+11, §5.19,5.20-23]. Slightly more sophisticated expositions can be found in [Sta99, Appendix 7.2] and [Mac95, Appendix I.A].

Definition 5.1. Given a complex n-dimensional vector space V, a *polynomial representation* (resp. *rational represenation* of $\mathrm{GL}(V)$ is a representation

$$\rho : \mathrm{GL}(V) \to \mathrm{GL}_N(\mathbb{C})$$

where the entries of $\rho(g)$ are polynomial (resp. rational) functions of the entries of g.

Remark 5.2. For those who know some algebraic geometry, the rational representations are rational maps on the ambient matrix ring $\mathbb{C}^{n^2}$, but they are regular on the open set $\mathrm{GL}_n(\mathbb{C})$. The polynomial representations are regular on the whole of $\mathbb{C}^{n^2}$. Watch out: some authors use "polynomial representation" synonymously with rational representations.

Example 5.3. If $V = \mathbb{C}^2$, then $\mathcal{S}^2(V) \cong \mathbb{C}^3$ has a natural action of $\mathrm{GL}(V)$. Indeed, choosing bases $\{x, y\}$ for V and $\{x^2, xy, y^2\}$ for $\mathcal{S}^2(V)$ we obtain a representation $\rho : \mathrm{GL}_2(k) \to \mathrm{GL}_3(k)$ with

$$\rho\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}$$

as explicit description.

Example 5.4. If V has dimension n, recall that $\Lambda^n(V)$ has dimension 1. The natural representation $\rho : \mathrm{GL}(V) \to \mathrm{GL}(\Lambda^n(V)) \cong k$ is the determinant.

More generally, $\mathcal{S}^k(V)$ and $\Lambda^k(V)$ have canonical actions of $\mathrm{GL}(V)$ for all $k \geq 0$.

Example 5.5. Consider the representation $\rho : \mathrm{GL}_1(\mathbb{C}) \to \mathrm{GL}_1(\mathbb{C})$ given by

$$\rho(z) = \frac{z}{|z|}.$$

This is neither a polynomial nor rational representation.

Example 5.6. The *dual representation* $V^\vee$ of V is the natural action of $\mathrm{GL}(V)$ on $V^\vee$. Choosing a basis and taking its dual basis, the dual representation corresponds to taking the transpose inverse $(A^T)^{-1}$ of a matrix A. This is not a polynomial representation since the entries of $(A^T)^{-1}$ have denominators. However, recall that the inverse matrix A^{-1} has the form $\det(A)\,\mathrm{adj}(A)$ where $\mathrm{adj}(A)$ is the *adjugate matrix*. The entries of the adjugate matrix are polynomial functions of the entries of the original matrix. Since $(A^T)^{-1} = \mathrm{adj}(A)^T \det(A)^{-1}$, we conclude that $V^\vee$ is a rational representation.

The difference between polynomial and rational representations is, fortunately, easily characterized. Indeed, the denominators that occur in the rational functions can only be powers of the determinant or else the representation is not defined for every point of $\mathrm{GL}(V)$.

Proposition 5.7. *If ρ is a rational representation of $\mathrm{GL}(V)$, then there is a unique minimal non-negative integer m and a unique polynomial representation σ such that such that*

$$\rho(g) = \frac{\sigma(g)}{\det(g)^m}$$

for all $g \in \mathrm{GL}(V)$.

Given a complex n-dimensional vector space V, recall that there is a natural action of S_d on the tensor power $V^{\otimes d}$ satisfying

$$\sigma(v_1 \otimes \cdots \otimes v_d) = \sigma(v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(d)})$$

for $\sigma \in S_d$. There is also a natural "diagonal" action of $\mathrm{GL}_n(V)$ on $V^{\otimes d}$ satisfying

$$g(v_1 \otimes \cdots \otimes v_d) = g(v_1) \otimes \cdots \otimes g(v_d)$$

for all $g \in \mathrm{GL}(V)$. These actions clearly commute, so we obtain an action of $S_d \times \mathrm{GL}(V)$ on $V^{\otimes d}$.

Definition 5.8. Given a partition $\lambda \vdash d$, we have the *Schur functor* $\mathbb{S}_\lambda$, which takes a vector space W to the following $\mathrm{GL}(W)$-representation:

$$\mathbb{S}_\lambda(W) := \mathrm{Hom}_{\mathbb{C}}^{S_d}(V_\lambda, W^{\otimes d}) \ .$$

Remark 5.9. For those who know some category theory, each Schur functor $\mathbb{S}_\lambda$ is a covariant functor from the category of finite-dimensional vector spaces to itself. This implies that if $f : W \to U$ is a linear map, then we also have a linear map

$$\mathbb{S}_\lambda(f) : \mathbb{S}_\lambda(W) \to \mathbb{S}_\lambda(U).$$

The fact that these are $\mathrm{GL}(W)$ representations can be seen as just a consequence of the fact that they are functors. (In particular, Schur functors can be defined for more general tensor categories.)

Theorem 5.10 (Schur-Weyl)**.** *Let W be an n-dimensional complex vector space and let $d \geq 0$. As an $S_d \times \mathrm{GL}(W)$-representation, there is a canonical decomposition*

$$W^{\otimes d} = \bigoplus_{\substack{\lambda \vdash d \\ \ell(\lambda) \leq n}} V_\lambda \otimes \mathbb{S}_\lambda(W)$$

where V_λ is the Specht module of λ and each

$$\mathbb{S}_\lambda(W) := \mathrm{Hom}_{\mathbb{C}}^{S_d}(V_\lambda, W^{\otimes d})$$

is an irreducible $\mathrm{GL}(W)$*-representation.*

Theorem 5.11. *The isomorphism classes of all* polynomial *irreducible representations of* $\mathrm{GL}(W)$ *are precisely the $\mathbb{S}_\lambda(W)$ where $\ell(\lambda) \leq d$. Moreover, if $g \in \mathrm{GL}(W)$ has eigenvalues $\alpha_1, \ldots, \alpha_n$, then*

$$\mathrm{tr}(\mathbb{S}_\lambda(g)) = s_\lambda(\alpha_1, \alpha_2, \cdots, \alpha_n, 0, 0, \cdots)$$

we conclude s_λ is the character of $\mathbb{S}_\lambda$.

By evaluating $\mathbb{S}_\lambda$ at the identity, we can determine the dimensions of the irreducible representations obtained from Schur functors.

Corollary 5.12 (Weyl Dimension Formula)**.** *If W is an n-dimensional vector space and λ is a partition. If $\ell(\lambda) \leq n$, then*

$$\dim\ \mathbb{S}_\lambda(W) = s_\lambda(1, 1, \ldots, 1, 0, \ldots) = \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

If $\ell(\lambda) > n$, then $\dim\ \mathbb{S}_\lambda(W) = 0$.

Example 5.13. Since $V_{(d)}$ is the trivial S_d-representation, we see that

$$\mathbb{S}_{(d)}(W) = \mathrm{Hom}_{\mathbb{C}}^{S_d}(\mathbb{C}, W^{\otimes d}) \cong \mathrm{Sym}^d(W) \cong \mathcal{S}^d(W).$$

Example 5.14. Since $V_{(1^d)}$ is the sign S_d-representation ϵ, we see that

$$\mathbb{S}_{(1^d)}(W)L = \mathrm{Hom}_{\mathbb{C}}^{S_d}(\epsilon, W^{\otimes d}) \cong \mathrm{Alt}^d(W) \cong \Lambda^d(W).$$

Note that $\mathbb{S}_{(1^d)}(W) = 0$ when $\dim(W) > d$.

References

[EGH+11] Pavel Etingof, Oleg Golberg, Sebastian Hensel, Tiankai Liu, Alex Schwendner, Dmitry Vaintrob, and Elena Yudovina. *Introduction to representation theory*, volume 59 of *Student Mathematical Library*. American Mathematical Society, Providence, RI, 2011. With historical interludes by Slava Gerovitch.

[FH91] William Fulton and Joe Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[Mac95] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

[Sta99] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.