Introduction to Representation Theory

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1 Basic Concepts

Some of the notions discussed here can also be found in [Ser77, §1]. We will use many other sources for this material, but most (correctly!) use the module-theoretic perspective, which we want to defer for the moment.

Definition 1.1. Let V be a vector space over a field k. The *general linear group of* V, denoted $\mathrm{GL}(V)$, is the group of invertible linear transformations $V \to V$. For a non-negative integer n, we use the notation $\mathrm{GL}_n(k) := \mathrm{GL}(k^n)$.

The group $\mathrm{GL}_n(k)$ can be canonically identified with the set of $n \times n$ invertible matrices with entries in k.

Definition 1.2. Suppose G is a group. A *linear representation of* G is a group homomorphism $\rho : G \to \mathrm{GL}(V)$ for a vector space V. The *degree* of a linear representation is the dimension of the corresponding vector space V. A linear representation ρ is *faithful* if ρ is injective as a map of sets.

We often refer to the "representation V", indicating its underlying vector space, rather than the homomorphism ρ. We will sometimes write ρ_g instead of $\rho(g)$ as it is easier to read. Later, we will simply write "g" for $\rho(g)$ when there is no danger of confusion.

Example 1.3. For any group G, the *zero representation* is the representation of degree 0 (it is unique up to isomorphism).

Example 1.4. For any group G, the *unit* or *trivial representation* is the representation

$$\rho : G \to \mathrm{GL}_1(k)$$

given by

$$\rho(g) = \mathrm{id}_k$$

for all $g \in G$.

Example 1.5. Let $G = \langle g \rangle$ be the cyclic group of order n. There is a representation

$$\rho : G \to \mathrm{GL}_1(\mathbb{C})$$

defined by

$$\rho(g^k) = e^{\frac{2\pi i}{n}k} \ .$$

Example 1.6. Let $D_8 = \langle s, r \mid s^2, r^4, (sr)^2 \rangle$ be the dihedral group of order 8. There is a representation

$$\rho : G \to \mathrm{GL}_2(k)$$

defined on generators by

$$\rho(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho(r) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and extended to other elements by using the fact that ρ must be a group homomorphism. Note that ρ is a *faithful* representation if and only if $\mathrm{char}(k) \neq 2$.

Example 1.7. For any group G with a left action on a set X, let V be the vector space with basis indexed by X:

$$V = \mathrm{span}\{e_x : x \in X\} \ .$$

Define the *permutation representation* corresponding to X as the representation

$$\rho : G \to \mathrm{GL}(V)$$

defined such that

$$\rho(g)(e_x) = e_{g(x)}$$

for all $g \in G$ and $x \in X$.

Example 1.8. For any group G, the *regular representation of* G is the permutation representation of the action of G on itself by left multiplication. In other words, V is the vector space with basis indexed by G:

$$V = \operatorname{span}\{e_g : g \in G\}$$

and the representation

$$\rho : G \to \mathrm{GL}(V)$$

is defined such that

$$\rho(g)(e_h) = e_{gh}$$

for all $g, h \in G$.

Example 1.9. Linear representations $\rho : \mathbb{Z} \to \mathrm{GL}_n(k)$ are completely determined by the image of $\rho(1)$ since $\rho(n) = \rho(1)^n$ for all $n \in \mathbb{Z}$. This gives a canonical bijection of linear representations of $\mathbb{Z}$ of degree n and $n \times n$ invertible matrices.

1.1 Equivariant Maps

Definition 1.10. Let V and W be vector spaces over the same field. Suppose

$$\rho : G \to \mathrm{GL}(V), \text{ and}$$
$$\rho' : G \to \mathrm{GL}(W)$$

are linear representations of the same group G. A linear map $\tau : V \to W$ is *G-equivariant* if

$$\rho'(g) \circ \tau = \tau \circ \rho(g)$$

for all $g \in G$. The two representations are *similar* or *isomorphic* if there exists an invertible G-equivariant linear map $\tau : V \to W$. Let $\mathrm{Hom}_k^G(V, W)$ be the subset of G-equivariant linear maps $f : V \to W$.

An archaic term for G-equivariant map is “intertwining operator,” but this is disappearing (thankfully in my view).

Example 1.11. If A and B are similar invertible matrices, then there exists an invertible matrix P such that $A = PBP^{-1}$. Recall that $A^n = PB^nP^{-1}$ for all integers n. In view of Example 1.9, we conclude that equivalence classes of linear representations of $\mathbb{Z}$ of degree n are in canonical bijection with similarity classes of $n \times n$ invertible matrices.

Example 1.12. Let $\rho : \mathbb{Z}/4\mathbb{Z} \to \mathrm{GL}_1(\mathbb{C})$ be the representation where $\rho(1) = (i)$ and let $\sigma : \mathbb{Z}/4\mathbb{Z} \to \mathrm{GL}_2(\mathbb{C})$ be the representation where

$$\sigma(1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Define $\rho'(g) = \rho(g^{-1})$ and $\sigma'(g) = \sigma(g^{-1})$ for all $g \in G$. By considering eigenvalues, we conclude that σ' and σ are equivalent, but ρ' and ρ are not.

Example 1.13. Let ρ be the 3-dimensional permutation representation associated to the action of S_3 on $\{1, 2, 3\}$. In the usual basis, we have

$$\rho(23) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \rho(123) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Suppose $\zeta \in k$ is a primitive cube root of unity. Via the change of basis matrix

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta^2 & \zeta \\ 1 & \zeta & \zeta^2 \end{pmatrix}$$

we have a new representation σ where

$$\sigma(23) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \sigma(123) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}.$$

Observe that ρ and σ are similar by construction.

1.2 Subrepresentations

Definition 1.14. Let $\rho : G \to \mathrm{GL}(V)$ be a linear representation. A subspace $W \subset V$ is *stable under the action of* G if $\rho_g(w) \in W$ for all $g \in G$, $w \in W$. Define $\rho^W : G \to \mathrm{GL}(W)$ by $\rho^W(g) := \rho(g)|_W$ for all $g \in G$. We say ρ^W is a *subrepresentation* of ρ.

We will also say simply "W is a subrepresentation of V" in the above situation.

If W is a subrepresentation of V then

$$\rho_g(v + w) = \rho_g(v) + \rho_g(w) \in \rho_g(v) + W$$

for any $v \in V$, $w \in W$, and $g \in G$. This gives rise to a well-defined *quotient representation* $\rho' : G \to \mathrm{GL}(V/W)$ on the quotient space V/W.

Definition 1.15. Given representations (V, ρ), (W, σ) of a group G, the *direct sum representation* $\rho \oplus \sigma$ is the representation with underlying space $V \oplus W$ given by

$$(\rho \oplus \sigma)_g(v, w) := (\rho_g(v), \sigma_g(v))$$

for all $v \in V$, $w \in W$, and $g \in G$.

Proposition 1.16. *Let V and W be representations of G, and let $f : V \to W$ be a G-equivariant map. Then* $\ker(f)$ *is a subrepresentation of V.*

Example 1.17. In the notation of Example 1.13, we see that V can be written as a direct sum $U \oplus W$ where

$$U = \operatorname{span}\left\{\left(1, 1, 1\right)^T\right\}$$

is a trivial representation, and

$$W = \operatorname{span}\left\{\left(1, \zeta, \zeta^2\right)^T, \left(1, \zeta^2, \zeta\right)^T\right\}$$

is a 2-dimensional faithful representation of S_3.

Example 1.18. Suppose $G = \langle g \rangle$ is a finite cyclic group. Recall that every matrix of finite order can be diagonalized over $\mathbb{C}$. Thus, every finite-dimensional complex representation of G is a direct sum of 1-dimensional representations.

Example 1.19. If V is a representation of G, define the *space of invariants*

$$V^G = \{v \in V \mid gv = v \text{ for all } g \in G\}.$$

Observe that V^G is a subrepresentation of V and it is a direct sum of trivial representations.

Definition 1.20. A linear representation V is *irreducible* if $V \neq 0$ and the only subrepresentations of V are 0 and V itself. Otherwise, the representation is *reducible.*

Definition 1.21. A linear representation V is *indecomposable* if $V \neq 0$ and V cannot be written as a direct sum $V = W_1 \oplus W_2$ where W_1, W_2 are non-zero subrepresentations. Otherwise, the representation is *decomposable.*

A decomposable representation is clearly reducible, but the converse may not hold as the next example demonstrates.

Example 1.22. Let $G = \mathbb{Z}/p\mathbb{Z}$ for a prime p and consider the two-dimensional representation ρ over $\mathbb{F}_p$ given by

$$\rho(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

A vector $\begin{pmatrix} a \\ b \end{pmatrix}$ spans a G-stable subspace only if $b = 0$. The subspace spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a proper non-zero subrepresentation of ρ, so ρ is reducible. However, this is the only subrepresentation so ρ cannot be written as a direct sum of proper subrepresentations. We conclude that ρ is indecomposable.

Theorem 1.23 (Krull-Schmidt)**.** *Suppose V is a finite-dimensional representation of a finite group G. All decompositions*

$$V \cong W_1 \oplus W_2 \oplus \cdots \oplus W_r$$

into indecomposable representations are unique up to isomorphism and reordering.

Proof. Deferred for now. We will prove this in more generality when we discuss modules. □

Remark 1.24. The Krull-Schmidt theorem fails for representations $G \to \mathrm{GL}(\mathbb{Z})$, so the fact we're working over a field is vital here. Unfortunately, counterexamples are rather subtle!

Theorem 1.25 (Schur's Lemma)**.** *If V and W are finite-dimensional irreducible representations and k is algebraically closed, then*

$$\mathrm{Hom}_k^G(V, W) = \begin{cases} 0 & \text{if } V \ncong W \\ k & \text{if } V \cong W. \end{cases}$$

Proof. Let $\phi : V \to W$ be a G-equivariant map. Since V is irreducible, either $\ker(\phi) = 0$ or $\ker(\phi) = V$. Since W is irreducible, either $\mathrm{im}(\phi) = 0$ or $\mathrm{im}(\phi) = W$. Thus, ϕ is either 0 or an isomorphism.

Thus, we are reduced to the case where $W = V$. Now $\mathrm{End}_k^G(V) = \mathrm{Hom}_k^G(V, W)$ is a k-algebra under composition. If $\phi \in \mathrm{End}_k^G(V)$ is non-zero, it must be an isomorphism. Thus ϕ^{-1} is also in $\mathrm{End}_k^G(V)$. We conclude that $\mathrm{End}_k^G(V)$ is a division algebra. The result now follows from Lemma 1.26. □

Lemma 1.26. *If k is an algebraically closed field, then every finite-dimensional division k-algebra D is isomorphic to k.*

Proof. Suppose $a \in D \setminus \{0\}$. Since D is finite-dimensional, multiplication by a has a minimal polynomial $m_a(t) \in k[t]$. If m_a does not have degree 1, then $m_a(t) = f(t)g(t)$ for non-constant polynomials $f, g \in k[t]$ since k is algebraically closed. The equation $0 = m_a(a) = f(a)g(a)$ forces either $f(a) = 0$ or $g(a) = 0$ since D is a division ring. But this contradicts minimality of the minimal polynomial. We conclude that $m_a(t) = t - \lambda$ for some $\lambda \in k$. Thus a is scalar multiplication by λ and thus is an element of k. □

Definition 1.27. A linear representation V is *completely reducible* if V is a direct sum of irreducible representations.

Suppose V is completely reducible. The Krull-Schmidt theorem tells us that there exist irreducible representations $W_1, \ldots, W_r$, which are pairwise non-isomorphic, such that

$$V = \bigoplus_{i=1}^{r} V_i \tag{1.1}$$

where each V_i is isomorphic to

$$V_i = W_i^{\oplus m_i}$$

for some positive integer m_i. The expression (1.1) is called the *isotypic decomposition of V*. The subrepresentations V_i are called the *isotypic components of V associated to W_i* while the integers m_i are called the *multiplicities of W_i in V*.

Each V_i is a *canonical* subrepresentation of V, while classifying subrepresentations isomorphic to W_i may depend on arbitrary choices. This generalizes how the eigenspaces of a linear transformation are canonical, while a basis of eigenvectors may depend on arbitrary choices.

Exercise 1.28. *Prove the Krull-Schmidt theorem for the case of completely reducible representations V when the field k is algebraically closed. (Hint: use Schur's Lemma.)*

The following theorem is the main reason why representation theory of finite groups in "good" characteristic is a completely different story from "bad" characteristic. By convention, every integer is "coprime to the characteristic of k" when the characteristic is 0.

Theorem 1.29 (Maschke)**.** *Let G be a finite group of order coprime to the characteristic of k. Every finite-dimensional representation of G is completely reducible.*

This theorem follows by induction and the following lemma:

Lemma 1.30. *Let G be a finite group of order coprime to the characteristic of k. If V is a representation of G and W is a subrepresentation of V, then there exists a subrepresentation U of V such that $V = W \oplus U$.*

Proof. Let $Q : V \to W$ be a projection from V onto the subspace W. The projection Q exists for linear algebraic reasons and is *not* necessarily G-equivariant. However, we will tweak it so that it *is* G-equivariant. Define $P : V \to W$ via

$$P(v) = \frac{1}{|G|} \sum_{g \in G} g\left(Q\left(g^{-1} v\right)\right)$$

for all $v \in V$. Note that the condition on the characteristic of the field is needed to divide by the order of the group.

We claim that $P : V \to W$ is a G-equivariant projection onto W. Since Q restricts to the identity on W and W is G-stable, we see that $g \circ Q \circ g^{-1}$ is a projection onto W for all $g \in G$. Thus, we have

$$\begin{aligned} P^2(v) &= \frac{1}{|G|^2} \sum_{g,h \in G} gQg^{-1}hQh^{-1}v \\ &= \frac{1}{|G|^2} \sum_{g,h \in G} hQh^{-1}v \\ &= \frac{1}{|G|} \sum_{h \in G} hQh^{-1}v = P(v) \end{aligned}$$

and conclude that P is a projection. Since $P|_W = \mathrm{id}_W$, we conclude it is a projection onto W.

Now we show that P is equivariant. This follows from a reindexing trick $j = h^{-1}g$ in the following:

$$\begin{aligned} P(hv) &= \frac{1}{|G|}\sum_{g\in G} gQg^{-1}(hv) \\ &= \frac{1}{|G|^2}\sum_{j\in G} hjQj^{-1}v \\ &= h\left(\frac{1}{|G|}\sum_{j\in G} jQj^{-1}v\right) = hP(v). \end{aligned}$$

The kernel of P is the desired G-stable complement U. □

1.3 Constructions

Suppose (V,ρ) and (W,σ) are finite-dimensional linear representations of a group G over a field k.

Let $\mathrm{Hom}_k(V,W)$ denote the vector space of linear transformations $\tau : V \to W$. Let $V^\vee$ be the vector space dual to V. Let $V \otimes W$ be the tensor product over k.

Proposition 1.31. *The vector space* $\mathrm{Hom}_k(V,W)$ *has a canonical structure of a linear representation* τ *where*

$$\tau_g(f) = \sigma(g) \circ f \circ \rho(g)^{-1}$$

for all $f : V \to W$ *and* $g \in G$.

The G-action defined above has the very useful property that the set of G-equivariant homomorphisms is precisely the set of invariants of the G-action on the set of homomorphisms. In other words,

$$\mathrm{Hom}_k^G(V,W) = \mathrm{Hom}_k(V,W)^G.$$

Proposition 1.32. *The dual space* $V^\vee$ *has a canonical structure of a linear representation* $\rho^\vee$ *where*

$$(\rho^\vee)_g(f) = f \circ \rho(g)^{-1}$$

for $f : V \to k$ *and* $g \in G$.

Note that the above proposition is a special case of the action on homomorphisms when $W = k$ has a trivial G-action.

Proposition 1.33. *The tensor product $V \otimes W$ has a canonical structure of a linear representation $\rho \otimes \sigma$ where*

$$(\rho \otimes \sigma)_g(v \otimes w) = \rho_g(v) \otimes \sigma_g(w)$$

for $v \in V$, $w \in W$ and $g \in G$.

Note that the above proposition agrees with the G-action one obtains from the canonical isomorphism

$$\mathrm{Hom}_k(W, V) \cong W^\vee \otimes V.$$

Exercise 1.34. *For a vector space V, determine the actions of G on $\mathcal{T}^d(V)$, $\mathcal{S}^d(V)$, $\Lambda^d(V)$, $\mathrm{Sym}^n(V)$, and $\mathrm{Alt}^n(V)$. Verify that the various canonical maps are equivariant.*

References

[Ser77] Jean-Pierre Serre. *Linear representations of finite groups*. Graduate Texts in Mathematics, Vol. 42. Springer-Verlag, New York-Heidelberg, 1977. Translated from the second French edition by Leonard L. Scott.