Induced Representations

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"The basics" of induced representations can be found in essentially every introductory character theory text. For example: [AB95, §16], [EGH$^+$11, §5.8-5.11], [FH91, §3.3], [Lan02, §XVIII.6–9], or [Ser77, §3.3]. However, we go a bit further then "the basics" here; see [Ser77, §7–10].

1 Group Actions

Let G be a group.

Definition 1.1. A (left) *G-set* is a set X with a (left) action of a finite group G. Given G-sets X and Y, a *morphism of G-sets* or *G-equivariant map* is a function $f : X \to Y$ such that $f(gx) = gf(x)$ for all $x \in X$ and $g \in G$. An *isomorphism of G-sets* is a G-equivariant bijection.

Let H be a subgroup of G. Recall that a *left coset of H in G* is a subset of G of the form

$$gH = \{gh \mid h \in H\}$$

for some $g \in G$. Recall that the left cosets partition G. We denote by G/H the set of left cosets of H in G. We do *not* expect G/H to have a group structure unless H is normal.

Proposition 1.2. *The set G/H has a canonical transitive left G-action.*

Proof. The action is defined via

$$g \cdot (aH) := (ga)H$$

for $g, a \in G$. Suppose $aH = bH$ for $a, b \in G$. Then $a = bh$ for some $h \in H$. Thus $ga = gbh$ for all $g \in G$. Thus $gaH = gbH$. We conclude that the action is well-defined. Checking the remaining two axioms for a group action are left as an exercise. □

Remark 1.3. The definition of the left action on G/H is more subtle than a superficial reading of the proof would suggest. We do *not* have an analogous right action on G/H since it is not well-defined. Asking that the right action be well-defined is equivalent to asking that H be a normal subgroup.

Exercise 1.4. *Let H, K be subgroups of G. The G-sets G/H and G/K are isomorphic if and only if H and K are conjugate subgroups of G.*

Recall that a G-set X is *transitive* if X is nonempty and, for all $x \in X$ and $y \in Y$, there exists a $g \in G$ such that $gx = y$.

Exercise 1.5. *Every G-set can be written uniquely as a disjoint union of transitive G-sets.*

Proposition 1.6. *Every non-empty transitive G-set X is isomorphic to G/H for some subgroup $H \subseteq G$.*

Proof. Since X is non-empty, we may select some $x \in X$. Let H be the stabilizer of x in G:

$$H = \mathrm{Stab}_G(x) = \{g \in G \mid gx = x\}.$$

We claim there is a unique G-equivariant function $f : X \to G/H$ such that $f(x) = H$. Indeed, since G acts transitively on X, the condition $f(gx) = gf(x)$ determines the value of f at every point of X. To see that this is well-defined, we simply observe that $f(gx) = f(g'x)$ is equivalent to $g^{-1}g'x = x$, which is equivalent to $g^{-1}g' \in H$. □

1.1 Right actions

Everything stated above for *left* group actions has a natural generalization to *right* group actions, which we mostly leave as an exercise for the reader. We use the notation $H\backslash G$ for the set of right cosets of H in G.

The following exercises are strongly recommended if you haven't thought about the distinction of left and right actions before:

Exercise 1.7. *If X is a* left *G-set and G is abelian, then there is a* right *action on X via $x \cdot g := gx$ for $g \in G$ and $x \in X$.*

Exercise 1.8. *If X is a* left *G-set, then we obtain a* right *action on X via $x \cdot g := g^{-1}x$ for $g \in G$ and $x \in X$.*

Exercise 1.9. *The function $f : G/H \to H\backslash G$ given by $f(gH) = Hg$ is a well-defined bijection if and only if H is normal in G.*

1.2 Double cosets

Let H, K be subgroups of G.

Definition 1.10. A *(H,K)-double coset of* G is a set of the form

$$HgK := \{hgk \mid h \in H, k \in K\}$$

for some $g \in G$. The set of double cosets of G is denoted $H\backslash G/K$.

There are several different equivalent perspectives on double cosets. Double cosets are the equivalence classes for the equivalence relation where $g \sim hgk$ for all $h \in H$ and $k \in K$. Double cosets are the orbits of the left H-action on G/K. Double cosets are the orbits of the right K-action on $H\backslash G$. Finally, double cosets are the orbits of the *left* action of $H \times K$ on G via $(h, k) \cdot g := hgk^{-1}$.

Each of these perspectives is useful in different contexts. An immediate consequence is that double cosets partition G; in other words, HgK and $Hg'K$ are either disjoint or equal and every element of G is contained in some double coset. Another consequence is that each double coset is both a union of right cosets of H in G and a union of left cosets of K in G.

Note that unlike ordinary cosets, double cosets are not guaranteed to all have the same size. However, their size can be easily computed from the orbit stabilizer theorem:

Exercise 1.11. *If G is finite, then*

$$|HgK| = \frac{|H||K|}{|H \cap gKg^{-1}|} = \frac{|H||K|}{|g^{-1}Hg \cap K|}.$$

2 Definition of Induced Representation

Let H be a subgroup of a finite group G.

Given a finite dimensional representation (V, ρ) of G, we define the *restriction to H* as the representation $(V, \rho|_H)$ where the underlying vector space is the same, but the group homomorphism $\rho|_H : H \to \mathrm{GL}(V)$ has domain H. We often write $\mathrm{Res}_H^G(V)$ or $V{\downarrow}_H^G$ to emphasize the H-action, even though the vector space is the same.

Induced representations are a method of going in the other direction. We begin with a representation of H and produce a representation of G. It is convenient to define induced representations via their universal property:

Definition 2.1. Suppose H is a subgroup of a finite group G and W is a finite dimensional representation of H. The *induced representation* is a representation $\mathrm{Ind}_H^G W$ of G together with an H-equivariant map $\iota : W \to \mathrm{Ind}_H^G W$ such that for any other representation V of G and H-equivariant map $\varphi : W \to V$, there exists a unique G-equivariant map $\psi : \mathrm{Ind}_H^G W \to V$ such that $\psi \circ \iota = \varphi$.

An alternative notation for $\mathrm{Ind}_H^G W$ is $W{\uparrow}_H^G$.

As usual, the universal property is economical to state and is useful theoretically, but it is less easy to see that it exists. On the other hand, an explicit construction may have various unimportant "implementation details" that vary from author to author and are not of fundamental importance. Nevertheless, a direct construction must be carried out and we will need them later.

Proposition 2.2. *Induced representations exist and are unique up to unique isomorphism.*

Proof. The uniqueness statement follows from the universal mapping property by abstract nonsense. The interesting part is existence.

We are given a finite group G with subgroup H and a representation (W, σ) of H. Choose a system of distinct representatives X for G/H. Let V be the direct sum $\bigoplus_{x \in X} W$. Labeling the xth summand of V by xW, we obtain isomorphisms $\phi_x : W \to xW$.

We now observe that

$$\mathrm{End}_k(V) = \bigoplus_{x \in X} \bigoplus_{y \in X} \mathrm{Hom}_k(xW, yW)$$

has a "block matrix" structure. Given an element $\psi \in \mathrm{End}_k(V)$, denote by $\psi^{y,x}$ the restriction to each $\mathrm{Hom}_k(xW, yW)$.

Fix $g \in G$; we will define a map $\rho_g \in \mathrm{End}_k(V)$ by defining each component $\rho_g^{y,x}$. For all $g \in G$ and $x, y \in X$, we define

$$\rho_g^{y,x} = \begin{cases} \phi_y \circ \sigma_{y^{-1}gx} \circ \phi_x^{-1} & \text{if } gxH = yH, \\ 0 & \text{otherwise,} \end{cases}$$

where we observe that $y^{-1}gx \in H$ so the evaluation of σ makes sense.

It is now tedious, but straightforward, to check that (V, ρ) is linear representation of G satisfying the desired universal property. □

Remark 2.3. Choose a system of distinct representatives X for G/H and write $\mathrm{Ind}_H^G W = \bigoplus_{x \in X} W$ as in the proof of Proposition 2.2. There is an H-equivariant linear map

$$\pi : \mathrm{Ind}_H^G W \to W$$

obtained by taking the sum

$$\pi\left((w_x)_{x \in X}\right) := \sum_{x \in X} w_x$$

over all x in a system of distinct representatives X. The map π turns out not to depend on the choice of X. and can be thought of as a generalization of the trace.

Defining this map gives a different universal property: given any representation V of G along with an H-equivariant map $\varphi : V \to W$, there exists a unique G-equivariant map $\psi : V \to \mathrm{Ind}_H^G W$ such that $\pi \circ \psi = \varphi$. In a more general context, objects satisfying the two universal properties may not coincide and we refer to those satisfying this second construction as *coinduced representations.*

3 Monomial Representations

In practice, many representations are induced from a subgroups. We've already seen many examples. In particular:

Proposition 3.1. *If V is a permutation representation of a finite group G, then V is a direct sum of representations induced from the trivial representation of subgroup.*

Proof. Recall that V has a basis $\{e_x\}_{x\in X}$ where X is a G-set. Since all G-sets are a disjoint union of transitive G-sets, we see that V is a direct sum of permutation representations corresponding to transitive G-sets. Thus, without loss of generality we may assume $X \cong G/H$ for some subgroup $H \subseteq G$.

Choosing a system of distinct representatives S for G/H, we now identify the direct summands sW in the proof of Proposition 2.2 with the spans of the basis elements $\{e_{sH}\}$ in V. One checks that the group action of G is identical in both cases. □

Example 3.2. The regular representation of G is precisely the representation $\operatorname{Ind}_1^G k$ obtained by induction from the trivial representation of the trivial group.

Definition 3.3. A representation (V, ρ) of a group G is a *monomial representation* or a *twisted permutation representation* if there exists a basis $\{e_1, \ldots, e_n\}$ of V such that, for all $1 \leq i \leq n$ and $g \in G$ we have

$$\rho_g(e_i) = \lambda e_j$$

for some $1 \leq j \leq n$ and $\lambda \in k^\times$.

Monomial representations are those with a basis such that the corresponding matrices have exactly one non-zero element in each row and column. Permutation representations are those where that non-zero element is always equal to one.

Exercise 3.4. *If V is a monomial representation of a finite group G, then V is a direct sum of representations induced from one-dimensional representations of subgroups.*

4 Characters of Induced Representations

In this section, we assume that the base field k is $\mathbb{C}$.

Definition 4.1. Given a class function $f : G \to \mathbb{C}$, define $\operatorname{Res}_H^G f : H \to \mathbb{C}$ as the class function obtained by restricting the domain to H.

It is clear that the character of $\operatorname{Res}_H^G V$ is $\operatorname{Res}_H^G \chi_V$.

Definition 4.2. Given a class function $f : H \to \mathbb{C}$, we construct a new class function $\operatorname{Ind}_H^G f : G \to \mathbb{C}$ by the formula

$$(\operatorname{Ind}_H^G f)(g) = \frac{1}{|H|} \sum_{\substack{a \in G \\ a^{-1}ga \in H}} f(a^{-1}ga)$$

for $g \in G$.

Proposition 4.3. *The character of* $\operatorname{Ind}_H^G V$ *is* $\operatorname{Ind}_H^G \chi_V$.

Proof. We use the block matrix notation from Proposition 2.2 and set X to be a system of distinct representatives for G/H. We want to compute the trace of ρ_g for some $g \in G$. This is equivalent to summing the traces of the elements $\rho_g^{x,x}$ for all $x \in X$. We observe that $\rho_g^{x,x} = 0$ unless $gx = xh$ for some $h \in H$. Thus we are only summing over $x \in X$ such that $x^{-1}gx \in H$. In this case, we have $\rho_g^{x,x} = \phi_x \circ \sigma_h \circ \phi_x^{-1}$, which has the same trace as σ_h where $h = x^{-1}gx$. Thus, we have the formula

$$(\operatorname{Ind}_H^G f)(g) = \sum_{\substack{x \in X \\ x^{-1}gx \in H}} f(x^{-1}gx).$$

Since f is a class function for H, we see that $f(x^{-1}gx) = f(y^{-1}gy)$ whenever $xH = yH$. Thus, summing over G instead of X amounts to multiplication by the size of the cosets $|H|$. □

Exercise 4.4. *Prove that the function* $\operatorname{Res}_H^G : C(G) \to C(H)$ *is a ring homomorphism.*

Exercise 4.5. *Prove that the function* $\operatorname{Ind}_H^G : C(H) \to C(G)$ *is an additive group homomorphism, but a ring homomorphism only if* $G = H$. *(Hint: what is* $(\operatorname{Ind}_H^G 1)(1)$*?)*

Proposition 4.6. *If* φ *is a class function of* H *and* ψ *is a class function of* G*, then* $\operatorname{Ind}_H^G((\operatorname{Res}_{\mathrm{H}}^{\mathrm{G}} \psi) \cdot \varphi) = \psi \cdot (\operatorname{Ind}_H^G \varphi)$.

Proof. Since $\psi(a^{-1}ga) = \psi(g)$ for all $a \in G$, we have

$$\frac{1}{|H|} \sum_{\substack{a \in G \\ a^{-1}ga \in H}} \psi(a^{-1}ga)\varphi(a^{-1}ga) = \psi(g) \left(\frac{1}{|H|} \sum_{\substack{a \in G \\ a^{-1}ga \in H}} \varphi(a^{-1}ga) \right)$$

for all $g \in G$. □

Corollary 4.7. *The image of* Ind_H^G *is an ideal of* $C(G)$.

The above facts are stated for the ring of class functions $C(G)$, but one can check that they also hold for the representation ring $R(G)$ as well.

5 Frobenius Reciprocity

Theorem 5.1. *Suppose* V *is a representation of* G *and* W *is a representation of a subgroup* H *of* G*. There is a canonical isomorphism*

$$\operatorname{Hom}_k^G(\operatorname{Ind}_H^G W, V) = \operatorname{Hom}_k^H(W, \operatorname{Res}_H^G V).$$

Proof. Recall that we have an H-equivariant map $\iota : W \to \operatorname{Ind}_H^G(W)$ by the definition of the induced representation. Thus, given an element $\psi \in \operatorname{Hom}_k^G(\operatorname{Ind}_H^G W, V)$, we obtain an element $\varphi \in \operatorname{Hom}_k^H(W, \operatorname{Res}_H^G V)$ by the composition $\varphi = \psi \circ \iota$. We uniquely recover ψ from φ from the universal mapping property, showing the two sets are in bijection. We've seen that composition of linear maps induces a linear transformation on Hom-spaces so we have the desired isomorphism of vector spaces. $\square$

An immediate corollary is the famous:

Theorem 5.2 (Frobenius Reciprocity)**.** *If* φ *is a class function of* H *and* ψ *is a class function of* G*, then*

$$(\operatorname{Ind}_H^G \varphi, \psi)_G = (\varphi, \operatorname{Res}_H^G \psi)_H$$

where we use subscripts to emphasize that one inner product is in $C(G)$ *while the other is in* $C(H)$*.*

Proof. In the case where φ and ψ are characters, we have

$$\begin{aligned}
&(\operatorname{Ind}_H^G \varphi, \psi)_G \\
=&\dim_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}}^G(\operatorname{Ind}_H^G W, V) \\
=&\dim_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}}^H(W, \operatorname{Res}_H^G V) \\
=&(\varphi, \operatorname{Res}_H^G \psi)_H.
\end{aligned}$$

This extends to all class functions by linearity since the characters are a spanning set. $\square$

6 Mackey decomposition

Here we learn how restriction and induction interact.

Theorem 6.1 (Mackey Decomposition)**.** *Let K, H be subgroups of G and suppose (W, σ) is a representation of H. Then*

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G W \cong \bigoplus_{[x]\in K\backslash G/H} \operatorname{Ind}_{H_x}^K W_x$$

where $H_x = xHx^{-1} \cap K$, and (W_x, σ_x) is a representation of H_x with underlying vector space W defined by $\sigma_x(g) = \sigma(x^{-1}gx)$ for all $g \in H_x$.

Proof. Let X be a system of distinct representatives for G/H. Observe that H_x is the stabilizer of the K-action on each $x \in X$. Let $X_1, \ldots, X_r$ be the subsets of X corresponding to the K-orbits on G/H. Observe that the sets $\{X_i\}$ correspond bijectively to the double cosets in $K\backslash G/H$.

We recall the explicit description of $\tau = \operatorname{Ind}_H^G \rho$ from Proposition 2.2 as a direct sum $\bigoplus_{x\in X} xW$, with isomorphisms $\phi_x : W \to xW$ and τ_g decomposing as

$$\rho_g^{y,x} = \begin{cases} \phi_y \circ \rho_{y^{-1}gx} \circ \phi_x^{-1} & \text{if } gxH = yH, \\ 0 & \text{otherwise,} \end{cases}$$

in each $\operatorname{Hom}_k(xW, yW)$ for $x, y \in X$ and $g \in G$.

Let W_i be the sum of the subspaces xW for $x \in X_i$. We have

$$\operatorname{Ind}_H^G W = \bigoplus_{i=1}^{r} W_i$$

where each W_i is easily seen to be K-stable. It remains to prove that $W_i \cong \operatorname{Ind}_{H_x}^K W_x$ for some choice of $x \in X_i$.

Again by Proposition 2.2, we have $\tau_i = \operatorname{Ind}_{H_x}^K \sigma_x$ with $\psi_y : W_x \to yW_x$ and

$$\rho_g^{z,y} = \begin{cases} \psi_z \circ \rho_{z^{-1}gy} \circ \psi_y^{-1} & \text{if } gyH_x = zH_x, \\ 0 & \text{otherwise,} \end{cases}$$

in each $\operatorname{Hom}_k(yW, zW)$ for $y, z \in X_i$ and $g \in K$.

For each $y \in X_i$, define $\pi_y : yW \to yW_x$ via $\pi_y = \psi_y \circ \phi_y^{-1}$. The direct sum $\bigoplus_{y\in X_i} \pi_y$ is an isomorphism from W_i to $\operatorname{Ind}_{H_x}^K \sigma_x$. One checks that this is K-equivariant as desired. □

We now turn to an application where $H = K$. As in the statement of the Mackey Decomposition Theorem, for any element $x \in G$, we define $H_x = xHx^{-1} \cap H$. Observe that while $\operatorname{Res}^H_{H_x} W$ and W_x are both representations of H_x, they are not isomorphic in general! If χ is the character for W in $C(H)$, let χ_x denote the character of W_x in $C(H_x)$.

Theorem 6.2 (Mackey Irreducibility Criterion)**.** *The representation* $\operatorname{Ind}^G_H W$ *is irreducible if and only if* W *is irreducible and, for every* $x \in G \setminus H$*, we have* $(\chi_x, \operatorname{Res}^H_{H_x} \chi) = 0$*.*

Proof. We use Frobenius Reciprocity and the Mackey Decomposition liberally:

$$\begin{aligned}
&(\operatorname{Ind}^G_H \chi, \operatorname{Ind}^G_H \chi)\\
=&(\chi, \operatorname{Res}^G_H \operatorname{Ind}^G_H \chi)\\
=&(\chi, \sum_{[x]\in H\backslash G/H} \operatorname{Ind}^H_{H_x} \chi_x)\\
=&\sum_{[x]\in H\backslash G/H} (\chi, \operatorname{Ind}^H_{H_x} \chi_x)\\
=&\sum_{[x]\in H\backslash G/H} (\operatorname{Res}^H_{H_x} \chi, \chi_x).
\end{aligned}$$

If $x \in H$, then they are equal and $\chi = \operatorname{Res}^H_{H_x} \chi = \chi_x$. Thus

$$(\operatorname{Ind}^G_H \chi, \operatorname{Ind}^G_H \chi) = (\chi, \chi) + \sum_{\substack{[x]\in H\backslash G/H\\ x\notin H}} (\operatorname{Res}^H_{H_x} \chi, \chi_x)$$

Since $\operatorname{Res}^H_{H_x} \chi$ and χ_x are actual characters of H_x, their inner product is always a non-negative integer. Moreover, (χ, χ) is a strictly positive integer. Thus $(\operatorname{Ind}^G_H \chi, \operatorname{Ind}^G_H \chi) = 1$ if and only if $(\chi, \chi) = 1$ and every other summand vanishes. The theorem follows. □

Corollary 6.3. *Suppose* H *is normal in* G*. The representation* $\operatorname{Ind}^G_H \rho$ *is irreducible if and only if* ρ *is irreducible and* $\rho_x \neq \rho$ *for all* $x \notin H$*.*

Proof. When H is normal, $H_x = H$ since $xHx^{-1} = H$ for all $x \in G$. In particular, $\operatorname{Res}^G_H(\rho) = \rho$. Assuming χ is irreducible, the condition $\big(\chi_x, \operatorname{Res}^G_H(\chi)\big) = 0$ is equivalent to $\chi_x \neq \chi$ and the result follows from the Mackey Irreducibility Criterion. □

7 Examples

7.1 Dihedral Groups

Let G be the dihedral group

$$D_{2n} = \langle r, s \mid s^2, r^n, (sr)^2 \rangle$$

and let H be the cyclic subgroup $\langle r \rangle$. Let (W, σ) be the 1-dimensional representation of H given by $\sigma(r) = (\zeta)$ where ζ is an nth root of unity.

We will construct $\operatorname{Ind}_H^G W$, which we call (V, ρ) for brevity. We use the notation from Proposition 2.2. Let $X = \{e, s\}$ be a set of distinct representatives for G/H. To determine ρ, we require understanding $\rho_g^{x,y}$ for generators $g = r, s$ and all pairs $x, y \in X$. After computing

$$r(eH) = eH,\ r(sH) = sH,\ s(eH) = sH,\ r(eH) = rH$$

we find

$$\rho(r) = \begin{pmatrix} \sigma(e^{-1}re) & 0 \\ 0 & \sigma(s^{-1}rs) \end{pmatrix} = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$$

and

$$\rho(s) = \begin{pmatrix} 0 & \sigma(e^{-1}ss) \\ \sigma(s^{-1}se) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

7.2 The Symmetric Group on 4 Letters

Let G be the symmetric group S_4 on 4 letters and let H be the subgroup generated by $(1\ 2)(3\ 4)$ and $(2\ 3)$. Observe that H is a non-normal subgroup of index 3, which is isomorphic to D_8. Let (W, σ) be the 1-dimensional representation of H given by $\sigma((1\ 2)(3\ 4)) = (-1)$ and $\sigma((2\ 3)) = (1)$. Note that the kernel of σ is generated by $(1\ 4)$ and $(2\ 3)$.

We construct the induced representation (V, ρ) using Proposition 2.2 as before. First, let us take $X = \{e, (1\ 2\ 3), (1\ 3\ 2)\}$ as our system of distinct representatives for G/H. We compute

$$(1\ 2)(3\ 4) \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
$$(2\ 3) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$(1\ 2\ 3) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

in this case.

7.3 A Non-Induced Representation

Here we develop a non-abelian example of a irreducible representation that is *not* an induced representation. I recommend only skimming this section on a first reading.

Recall that *Hamilton's quaternion algebra* $\mathbb{H}$ is the division $\mathbb{R}$-algebra with basis $\{1, i, j, k\}$ and multiplication determined by $ij = k$, $jk = i$, $ki = j$ and $i^2 = j^2 = k^2 = -1$. We have an embedding $\mathbb{H} \hookrightarrow \mathrm{M}_2(\mathbb{C})$ by extending

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad i \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad k \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

by linearity. This embedding is a homomorphism of $\mathbb{R}$-algebras. (Note that there is no embedding of $\mathbb{H}$ into the *real* 2×2 matrices.)

A general element x of $\mathbb{H}$ of the form

$$x = a + bi + cj + dk$$

where $a, b, c, d \in \mathbb{R}$. We define the *conjugate* of x as

$$\overline{x} = a - bi - cj - dk,$$

and the *norm* of x as

$$\|x\| = \sqrt{\overline{x}x} = \sqrt{a^2 + b^2 + c^2 + d^2},$$

which is a non-negative real number. We see that $\mathbb{H}$ is a division algebra since $x^{-1} = \overline{x}/\|x\|$ is an explicit formula for the inverse when $x \neq 0$.

A general element x of $\mathbb{H}$ is a *Lipschitz quaternion* if $a, b, c, d \in \mathbb{Z}$. The element x is a *Hurwitz quaternion* if either a, b, c, d are all integers, or all half-integers. The set L of Lipschitz quaternions and the set H of Hurwitz quaternions are both subrings of Hamilton's quaternions. The unit groups for these rings can be found by observing that the norm of an invertible element must be 1.

The *quaternionic group* $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ is the unit group $U(L)$ of the ring L. It is evident from the embedding $\mathbb{H} \to \mathrm{M}_2(\mathbb{C})$ described above that Q_8 has a representation by "twisted permutation matrices." Namely, there is a choice of basis where the group acts by permuting the basis elements and multiplying them by scalars.

The unit group $U(H)$ of the ring H contains Q_8 and the 16 elements

$$\frac{\pm 1 \pm i \pm j \pm k}{2}$$

where all possible sign combinations are permitted. One checks that $\frac{1}{2}(1 + i + j + k)$ has order 6, and thus we have a representation with image

$$\left\langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1+i & 1+i \\ -1+i & 1-i \end{pmatrix} \right\rangle.$$

Exercise 7.1. *Prove that $U(H)$ does not have a subgroup of index 2.*

An immediate consequence of this exercise is that $U(H)$ cannot be induced from a 1-dimensional representation.

Recall that A_4 is often called the "tetrahedral group" since it is isomorphic to the set of orthogonal matrices preserving a regular tetrahedron. Consequently, the group $U(H)$ is sometimes called the "binary tetrahedral group" due to the following exercise.

Exercise 7.2. *Let Z be the center of $U(H)$. Prove that the center Z of $U(H)$ has order 2 and $U(H)/Z$ is isomorphic to A_4.*

Exercise 7.3. *Prove that $U(H)$ is isomorphic to $Q_8 \rtimes (\mathbb{Z}/3\mathbb{Z})$.*

Exercise 7.4. *Prove that $U(H)$ is isomorphic to the special linear group $\mathrm{SL}_2(\mathbb{F}_3)$ over the field of 3 elements.*

8 Applications

Suppose G is a finite group and $\phi : G \to G$ is a group automorphism. For any class function χ, define the class function $\phi \cdot \chi := \chi \circ \phi^{-1}$.

Exercise 8.1. *Prove the following. The assignment $\chi \mapsto \phi \cdot \chi$ induces an action of $\mathrm{Aut}(G)$ on $C(G)$. The action preserves both the ring and inner product space structure on G, takes characters to characters, and irreducible characters to irreducible characters. The subgroup $\mathrm{Inn}(G)$ of inner automorphisms is in the kernel of the action.*

Lemma 8.2. *Suppose N is a normal subgroup of G and V is an irreducible representation of G. One of the following holds:*

1. $\mathrm{Res}_N^G V$ *is isotypic, or*
2. $V \cong \mathrm{Ind}_H^G W$ *where W is an irreducible representation of a proper subgroup H of G such that $N \subseteq H$.*

Proof. Consider the isotypic decomposition

$$Res_N^G V \cong U_1 \oplus U_2 \oplus \cdots \oplus U_r$$

where $U_1, \ldots, U_r$ are the distinct isotypic subrepresentations of N. If $r = 1$, then we are in the first case; thus we may assume $r \geq 2$. For any $g \in G$ and index i, we see that $g(U_i)$ is an N-stable subspace of V; indeed, for every $n \in N$ we have some $n' \in N$ such that $ngU_i = gn'U_i = gU_i$. Let H be the set of $g \in G$ such that $g(U_1) = U_1$. Thus U is an H-subrepresentation of V. By the universal mapping property, there is a G-equivariant morphism $\mathrm{Ind}_H^G U_1 \to V$. Since G acts transitively on the U_i's, we obtain an isomorphism. $\square$

Definition 8.3. A finite group G is *supersolvable* if there exists a chain of subgroups

$$1 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_r = G$$

where each G_i is normal in G and G_{i+1}/G_i is cyclic for each $i = 0, \ldots, r$.

Exercise 8.4. *Prove the following implications for a finite group G:*

$$p\text{-group} \implies \text{nilpotent} \implies \text{supersolvable} \implies \text{solvable},$$

and find explicit examples showing the converses may not hold.

Exercise 8.5. *All subgroups and quotient groups of a supersolvable group are supersolvable.*

Lemma 8.6. *If G is a nonabelian supersolvable group, then there exists a normal abelian subgroup A of G such that A is not contained in the center of G.*

Proof. Let Z be the center of G. Then G/Z is supersolvable and there exists a normal cyclic subgroup $C \subseteq G/Z$. Let A be the preimage of C via the quotient map $G \to G/Z$. Since C is normal in G/Z, we see A is normal in G. Since A is a central extension with cyclic quotient, it must be abelian. □

Theorem 8.7. *Every representation of a supersolvable group is monomial.*

Proof. We proceed by induction on the order of the group G. We may assume without loss of generality that our representation (V, ρ) is irreducible. Since quotients of supersolvable groups are supersolvable, by induction we may assume that V is a faithful representation. If G is abelian, then we are done. Thus, we may assume G is nonabelian and, by Lemma 8.6, there is a normal abelian subgroup A of G not contained in the center.

We claim $\operatorname{Res}_A^G V$ is *not* isotypic. An isotypic representation of an abelian group consists of scalar matrices. Since these commute with every other matrix in the group, and the representation is faithful, this would mean A was in the center of G, a contradiction. Thus, we may appeal to Lemma 8.2 to conclude V is induced from a proper subgroup H. Since H is supersolvable, by the induction hypothesis we conclude that V is induced from a monomial representation. Thus V is monomial as desired. □

9 Semidirect Products

Throughout this section $G = A \rtimes H$ where A is a finite abelian group and H is a finite group.

Recall that $A^\vee := \operatorname{Hom}_{\text{grp}}(A, \mathbb{C}^\times)$ is an abelian group that is isomorphic to A itself (although not canonically). There is a left action of G on $A^\vee$ via

$$(g\chi)(a) = \chi(g^{-1}ag)$$

where $g \in G$, $\chi \in A^\vee$ and $a \in A$. Since A is abelian, the G-action factors through the projection onto the subgroup H.

Let $\chi_1, \ldots, \chi_r$ be a system of distinct representatives for the H-orbits of $A^\vee$. Define $H_i = \mathrm{Stab}_H(\chi_i)$. In particular,

$$A^\vee \cong H/H_1 \sqcup \cdots \sqcup H/H_r$$

as H-sets.

Given an irreducible representation ρ of H_i, we will define a representation $\theta_{i,\rho}$ as follows. Let $G_i = A \rtimes H_i$. Extend ρ to a representation $\widetilde{\rho}$ of G_i by composition with the map $G_i \to H_i$. Extend χ_i to a representation $\widetilde{\chi_i}$ of G_i by defining $\widetilde{\chi_i}(ah) = \chi_i(a)$ for all $a \in A$ and $h \in H_i$; this makes sense because H_i acts trivially on χ_i. Now finally, we define

$$\theta_{i,\rho} := \mathrm{Ind}_{G_i}^G \widetilde{\chi_i} \otimes \widetilde{\rho},$$

which is a representation of G.

Theorem 9.1. *Every irreducible representation of G is isomorphic to some $\theta_{i,\rho}$, every $\theta_{i,\rho}$ is irreducible, and $\theta_{i,\rho} \cong \theta_{i',\rho'}$ if and only if $i = i'$ and $\rho \cong \rho'$.*

Proof. First we show that $\theta_{i,\rho}$ is irreducible. Let $\eta = \widetilde{\chi_i} \otimes \widetilde{\rho}$ where $\theta_{i,\rho} = \mathrm{Ind}_{G_i}^G \eta$. Let $x \in G \setminus G_i$ and $K_x = G_i \cap xG_ix^{-1}$. Observe that $\mathrm{Res}_A^{K_x} \eta = n\chi_i$ where n is the degree of η. As in the statement of the Mackey Decomposition, we define a representation of K_x via $\eta_x(g) = \eta(x^{-1}gx)$ for all $g \in K_x$. Now $\mathrm{Res}_A^{K_x} \eta = n\chi'$ where $\chi'(a) = \chi_i(x^{-1}ax)$ for all $a \in A$. Since $\chi_i(a) \neq \chi_i(x^{-1}ax)$ for some $a \in A$ we conclude that $\chi_i \neq \chi'$. Since χ_i and χ' are irreducible, we conclude that $(\chi_i, \chi') = 0$. Thus $(\eta, \eta_x) = 0$ and we conclude that $\theta_{i,\rho}$ is irreducible by Mackey's Irreducibility Criterion.

Now we show that the isomorphism class of $\theta_{i,\rho}$ is determined by i and ρ. The restriction $\mathrm{Res}_A^G \theta_{i,\rho}$ is a direct sum of characters $\nu_1, \ldots, \nu_s$ in $A^\vee$. From the Mackey Decomposition Theorem, we know that each ν_j is of the form $g \cdot \chi_i$ for some $g \in G$. Since A is abelian, each ν_j is of the form $h \cdot \chi_i$ for $h \in H$. In particular, all ν_j are from the same H-orbit of $A^\vee$, so the index i is determined. Let W be the isotypic component of $\mathrm{Res}_A^G V$ associated to the representation χ_i. The group G_i leaves W invariant and we see that W is isomorphic to the representation $\widetilde{\chi_i} \otimes \widetilde{\rho}$, which determines ρ.

Finally, we show that every irreducible representation of G has the desired form. Let (V, σ) be an irreducible representation of G. Let

$$\mathrm{Res}_A^G V = \bigoplus_{\chi \in A^\vee} W_\chi$$

be the isotypic decomposition. Pick a character $\chi_i \in A^\vee$ with $W_{\chi_i} \neq 0$ and let $H_i = \mathrm{Stab}_H(\chi_i)$. Observe that W_{χ_i} is a subrepresentation of V for the group H_i; let τ be this H_i-representation. We see that the G_i-action on W_{χ_i} is the representation $\widetilde{\chi}_i \otimes \widetilde{\tau}$. Using Frobenius Reciprocity, we have the sequence of inequalities

$$0 < \left(\widetilde{\chi}_i \otimes \widetilde{\tau}, \mathrm{Res}^G_{G_i} \sigma\right)_{G_i} = \left(\mathrm{Ind}^G_{G_i} \widetilde{\chi}_i \otimes \widetilde{\tau}, \sigma\right)_G \leq 1$$

where the first inequality follows from the fact that W_{χ_i} is a summand of V and the second inequality follows from the fact that σ is irreducible. We conclude that $\sigma \cong \theta_{i,\tau}$ as desired. □

10 Artin and Brauer's Theorem

Let $\mathcal{F}$ be a family of subgroups of a finite group G. Define the map

$$\Sigma_{\mathcal{F}} \mathrm{Ind} : \bigoplus_{H \in \mathcal{F}} R(H) \to R(G)$$

via

$$(\chi_H)_{H \in \mathcal{F}} \mapsto \sum_{H \in \mathcal{F}} \mathrm{Ind}^G_H \chi_H.$$

Theorem 10.1 (Artin's Theorem)**.** *If $\mathcal{F}$ is the set of cyclic subgroups of G, then $\Sigma_{\mathcal{F}} \mathrm{Ind}$ has finite cokernel.*

Proof. First, let V be a complex inner product space and consider the maps

$$\Sigma : \bigoplus_{i=1}^n V \to V \text{ and } \Delta : V \to \bigoplus_{i=1}^n V$$

defined via

$$\Sigma\left(v_1, \ldots, v_n\right) = \sum_{i=1}^n v_i \text{ and } \Delta(v) = (v, \ldots, v).$$

They are adjoint linear operators in the sense that

$$(\Sigma(w), v) = (w, \Delta(v))$$

for all $w \in \bigoplus_{i=1}^n V$ and $v \in V$.

Now, the function

$$\left(\bigoplus_{H\in\mathcal{F}} \operatorname{Res}_H^G\right) \circ \Delta : C(G) \to \bigoplus_{H\in\mathcal{F}} C(H)$$

is injective since class functions are determined by their values on cyclic subgroups. If a linear transformation is injective, then its adjoint operator

$$\Sigma_{\mathcal{F}} \operatorname{Ind} : \bigoplus_{H\in\mathcal{F}} C(H) \to C(G)$$

is surjective.

Now the linear map

$$\Sigma_{\mathcal{F}} \operatorname{Ind} : \mathbb{Q} \otimes_{\mathbb{Z}} \left(\bigoplus_{H\in\mathcal{F}} R(H)\right) \to \mathbb{Q} \otimes_{\mathbb{Z}} R(G)$$

must be surjective since tensoring with $\mathbb{C}$ produces a surjective linear map. The restriction to $\bigoplus_{H\in\mathcal{F}} R(H)$ therefore has finite cokernel, since the image is a subgroup of $R(G)$ of maximal rank. □

Definition 10.2. A group H is *p-elementary* for a prime p if $H \cong P \times C$ where P is a p-group and C is a cyclic group of order coprime to p. We say a group H is *elementary* if it is p-elementary for some prime p.

Theorem 10.3 (Brauer's Theorem)**.** *If $\mathcal{F}$ is the set of elementary subgroups of G, then $\Sigma_{\mathcal{F}} \operatorname{Ind}$ is surjective.*

Proof. See [Ser77, §10]. □

Corollary 10.4. *Every character is an integer linear combination of monomial characters.*

Proof. Brauer's Theorem tells us that every character is an integer linear combination of characters induced from elementary subgroups. An elementary group is nilpotent, thus supersolvable. By Theorem 8.7, all of their representations are monomial. A representation induced from a monomial representation is monomial. □

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