This is a **bonus** assignment. It is possible to get 100% in the course from only the first five assignments.

This assignment is “out of” 100 points, but there are far more than 100 points available. At the instructor’s discretion some “overflow” above 100 may be counted towards your final grade at the end of the course, but you should not expect this.

You are **not** expected to write up a full solution to every problem, but you **are** expected to at least think about every problem. Writing up every single problem on the assignment is probably not a good use of your time.

Throughout, the notation $Q_{a,b}$ denotes the quaternion k-algebra

$$Q_{a,b} = k\langle i, j \mid i^2 = a, j^2 = b, ij = -ij\rangle$$

where k is a field and a, b are non-zero elements of k.

Errata (2023/04/18): Fixed typo in 3.

Problem 1 (20 points) Prove that if R is a ring such that every left R-module is free, then R is a division algebra.

Problem 2 (20 points) Classify the two-sided ideals of $\mathrm{M}_2(\mathbb{Z})$.

Problem 3 (40 points) Determine the primes p such that $Q_{-1,p}$ splits over $k = \mathbb{Q}$.

Problem 4 Let k be a field and a, b, c, d non-zero elements of k.

- **(a)** (20 points) Find an explicit bijection $Q_{1,1} \cong \mathrm{M}_2(k)$.
- **(b)** (10 points) Prove that $Q_{a,b} \cong Q_{b,a}$.
- **(c)** (10 points) Prove that $Q_{a,b} \cong Q_{ac^2,bd^2}$.
- **(d)** (30 points) Prove that $Q_{1,b} \cong Q_{1,1}$.
- **(e)** (30 points) Prove that $Q_{a,1-a} \cong Q_{1,1}$.

Problem 5 In each of the following cases, determine the Wedderburn decomposition of the group ring kG (in other words, write kG as a product of simple k-algebras.

- **(a)** (20 points) The group ring $\mathbb{Q}C_3$ where C_3 is the cyclic group of order 3.
- **(b)** (20 points) The group ring $\mathbb{Q}S_3$ where S_3 is symmetric group on 3 letters.
- **(c)** (30 points) The group ring $\mathbb{Q}D_8$ where D_8 is the dihedral group on 4 letters.
- **(d)** (40 points) The group ring $\mathbb{Q}S_4$ where S_4 is symmetric group on 4 letters.
- **(e)** (30 points) The group ring $\mathbb{Q}Q_8$ where Q_8 is quaternionic group of order 8.
- **(f)** (80 points) The group ring $\mathbb{Q}G$ where $G = (\mathbb{Z}/3\mathbb{Z}) \rtimes_\phi (\mathbb{Z}/4\mathbb{Z})$ where $\phi : \mathbb{Z}/4\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/3\mathbb{Z})$ is the unique non-trivial homomorphism.