This assignment is "out of" 100 points, but there are far more than 100 points available. At the instructor's discretion some "overflow" above 100 may be counted towards your final grade at the end of the course, but you should not expect this.

You are **not** expected to write up a full solution to every problem, but you **are** expected to at least think about every problem. Writing up every single problem on the assignment is probably not a good use of your time.

Problem 1 (20 points) Let k be a commutative ring and G a finite group. Prove that the group algebra kG is commutative if and only if G is abelian.

Problem 2 (30 points) Determine the primitive central idempotents of $\mathbb{C}D_6$ where $D_6 = \langle s, r \mid s^2, r^3, (sr)^2 \rangle$ is the dihedral group of order 6.

Problem 3 Let $Z(\mathbb{Z}D_8)$ be the center of the group ring $\mathbb{Z}D_8$ where $D_8 = \langle s, r \mid s^2, r^4, (sr)^2 \rangle$ is the dihedral group of order 8.

(a) (10 points) Determine a basis for $Z(\mathbb{Z}D_8)$ as a free abelian group.

(b) (30 points) Determine the multiplication table of $Z(\mathbb{Z}D_8)$ in terms of this basis.

(c) (30 points) Determine the central idempotents of $\mathbb{Z}D_8$.

Problem 4 Here we consider idempotents group algebras kG for the cyclic group $G = \langle r | r^5 \rangle$.

(a) (10 points) Explicitly determine the primitive idempotents of $\mathbb{C}G$.

(b) (10 points) Explicitly determine the primitive idempotents of $\mathbb{R}G$.

(c) (10 points) Explicitly determine the primitive idempotents of $\mathbb{Q}G$.

(d) (20 points) Explicitly determine the primitive idempotents of $\mathbb{Z}G$.

(e) (10 points) Explicitly determine the primitive idempotents of $\mathbb{F}_2G$.

(f) (10 points) Explicitly determine the primitive idempotents of $\mathbb{F}_5G$.

Problem 5 Let k be a field, A be a k-algebra, and M be a left A-module. For a positive integer n, let M^n denote the column vectors with n entries in M viewed as a left $\mathrm{M}_n(A)$-module. (We do not assume that A or M is semisimple.) Here we explore ideas related to the *Morita-equivalence* of A and $\mathrm{M}_n(A)$.

(a) (30 points) Prove that a left A-module M is simple if and only if M^n is simple as a left $\mathrm{M}_n(A)$-module.

(b) (30 points) Prove that if M, N are left A-modules, then $\mathrm{Hom}_A(M, N) \cong \mathrm{Hom}_{\mathrm{M}_n(A)}(M^n, N^n)$.

(c) (30 points) Prove that every left $\mathrm{M}_n(A)$-module is isomorphic to M^n for some left A-module M.

Problem 6 (40 points) Let k be a field. Suppose H is a subgroup of a finite group G and W is a left kH-module. Prove that

$$kG \otimes_{kH} W \cong \mathrm{Hom}_{kH}(kG, W)$$

as left kG-modules.

Problem 7 Let k be a field and let A and B be k-algebras. We define a k-algebra structure on $A \otimes_k B$ via the rule

$$(a \otimes b) \cdot (a' \otimes b') := (aa') \otimes (bb')$$

extended by linearity.

(a) (20 points) Prove that the multiplication is well-defined and makes $A \otimes_k B$ a k-algebra.

(b) (20 points) Prove that $\mathrm{M}_n(k) \otimes_k \mathrm{M}_m(k) \cong \mathrm{M}_{mn}(k)$ as k-algebras.

(c) (30 points) Prove that the center $Z(A \otimes_k B)$ is isomorphic to the tensor product of the centers $Z(A) \otimes_k Z(B)$.

(d) (30 points) Prove that there is a canonical algebra homomorphism $A^{\mathrm{op}} \otimes_k A \to \mathrm{End}_k(A)$, which is an isomorphism when $A \cong \mathrm{M}_n(k)$.

Problem 8 (30 points) Let $R = R_1 \times \cdots \times R_n$ be a product of rings. Prove that the two-sided ideals of R are precisely the subsets $J_1 \times \cdots \times J_n$ where each J_i is a two-sided ideal of R_i.