This assignment is “out of” 100 points, but there are far more than 100 points available. At the instructor’s discretion some “overflow” above 100 may be counted towards your final grade at the end of the course, but you should not expect this.

You are **not** expected to write up a full solution to every problem, but you **are** expected to at least think about every problem. Writing up every single problem on the assignment is probably not a good use of your time.

Conventions: Below, $m_\lambda, e_\lambda, h_\lambda, p_\lambda, s_\lambda$ denote respecticely the monomial, elementary, complete homogeneous, power sum, and Schur bases for the ring of symmetric functions (or symmetric polynomials).

Errata (2023/03/18): Fixed typo in problem 1.

Errata (2023/03/23): Fixed error in problem 4.

Problem 1 (30 points) Let $p(n)$ denote the number of partitions of the non-negative integer n. Prove that

$$\sum_{n=0}^{\infty} p(n)t^n = \prod_{k=1}^{\infty} \frac{1}{1-t^k}$$

as formal power series (in other words, you do not need to worry about convergence).

Problem 2 Given a partition λ, recall that

$$z_\lambda = \prod_{i\geq 1} i^{m_i} m_i!$$

where $m_i = m_i(\lambda)$ denotes the multiplicities and

$$\epsilon_\lambda = (-1)^{|\lambda|-\ell(\lambda)}$$

where $|\lambda|$ is the weight of λ and $\ell(\lambda)$ is the length. Let $\sigma \in S_n$ be a permutation with cycle type λ where $n = |\lambda|$.

(a) (30 points) Prove that the centralizer $Z_{S_n}(\sigma)$ of σ in the group S_n has order z_λ.

(b) (10 points) Prove that the number of elements of S_n with cycle type λ is $n!/z_\lambda$.

(c) (20 points) Prove $\epsilon_\lambda = \operatorname{sgn}(\sigma)$ where $\operatorname{sgn} : S_n \to \{\pm 1\}$ is the sign morphism.

Problem 3 Let Δ be the Vandermonde determinant in the variables $x_1, \ldots, x_n$.

(a) (20 points) Find an explicit expression for Δ^2 in terms of elementary symmetric polynomials in the case where $n = 2$.

(b) (50 points) Find an explicit expression for Δ^2 in terms of elementary symmetric polynomials in the case where $n = 3$.

Problem 4 (30 points) Prove that $\sum_{\sigma\in S_n} x^{\sigma(\lambda)} = Cm_\lambda$ where

$$C = m_1(\lambda)! \cdots m_r(\lambda)!(n-\ell(\lambda))!$$

for a partition λ of length $r = \ell(\lambda) \leq n$.

Problem 5 (30 points) Prove that $s_{(d)} = h_d$ and $s_{(1)^d} = e_d$ for all positive integers d.

Problem 6 (30 points) Prove that

$$h_d = \sum_{\lambda \vdash d} \frac{p_\lambda}{z_\lambda}$$

and

$$e_d = \sum_{\lambda \vdash d} \epsilon_\lambda \frac{p_\lambda}{z_\lambda}$$

for all positive integers d.

Problem 7 (30 points) Determine the Kostka number $K_{\lambda\mu}$ for $\lambda = 41$ and $\mu = 2111$.

Problem 8 (30 points) Find explicit expressions for p_{111}, p_{21}, and p_3 in the elementary symmetric function basis.

Problem 9 (30 points) Recall that the *standard representation* V of S_n is the permutation representation obtained from the action of S_n on the set $\{1, \ldots, n\}$. Write V as a direct sum of Specht modules V_λ.

Problem 10 (30 points) Determine the values of the character χ_λ of the Specht module V_λ for $\lambda = 32$.

Problem 11 (50 points) The group $\mathrm{GL}_n(\mathbb{C})$ acts on the space $M_n(\mathbb{C})$ of $n \times n$-matrices by conjugation. Let $\mathfrak{sl}_n(\mathbb{C})$ be the subspace of $M_n(\mathbb{C})$ consisting of matrices with trace 0. Prove that $\mathfrak{sl}_n(\mathbb{C})$ is an irreducible rational subrepresentation. Determine the partition λ and the smallest non-negative integer m such that $\mathfrak{sl}_n(\mathbb{C}) \cong \det^m \otimes \mathbb{S}_\lambda(\mathbb{C}^n)$ for a Schur functor $\mathbb{S}_\lambda$.

Problem 12 (50 points) Let k be a field of characteristic $p > 0$ and let V be a k-vector space with basis $\{x_1, \ldots, x_n\}$. Prove that

$$W := \mathrm{span}_k\{x_1^p, \ldots x_n^p\}$$

is a $\mathrm{GL}(V)$-stable subspace of the space of symmetric polynomials $\mathcal{S}^p(V)$. (Thus the representation theory of $\mathrm{GL}(V)$ is *very* different when the characteristic is not 0.)

Problem 13 (100 points) (Theorem of Clark-Cooper) Let $f(x) = \sum_{i=0}^n c_i x^{n-i}$ be a monic polynomial over $\mathbb{C}$ with exactly k distinct roots $r_1, \ldots, r_k$. Prove that the values of $c_1, \ldots, c_k$ and $r_1, \ldots, r_k$ uniquely determine the multiplicities of the roots. Equivalently, they determine the remaining coefficients $c_{k+1}, \ldots, c_n$. (Hint: compare the power sums of all roots with the power sums of distinct roots.)