This assignment is "out of" 100 points, but there are far more than 100 points available. At the instructor's discretion some "overflow" above 100 may be counted towards your final grade at the end of the course, but you should not expect this.

You are **not** expected to write up a full solution to every problem, but you **are** expected to at least think about every problem. Writing up every single problem on the assignment is probably not a good use of your time.

Problem 1 (30 points) Let G be a finite group. Given a complex character χ of G, the *kernel of* χ is the subset

$$\ker(\chi) = \{g \in G \mid \chi(g) = \chi(1)\} \ .$$

Let $\chi_1, \ldots, \chi_n$ be the irreducible characters of G. Prove that every normal subgroup N of G can be written as

$$N = \bigcap_{i \in I} \ker(\chi_i)$$

for some subset $I \subseteq \{1, \ldots, n\}$.

Problem 2 (30 points) Find explicit examples showing that the functions $\mathrm{Res}_H^G : R(G) \to R(H)$ and $\mathrm{Ind}_H^G : R(H) \to R(G)$ are neither injective nor surjective in general.

Problem 3 (30 points) Describe explicitly the maps

$$\mathrm{Res}_{S_3}^{S_4} : R(S_4) \to R(S_3)$$
$$\mathrm{Ind}_{S_3}^{S_4} : R(S_3) \to R(S_4)$$

as matrices using the bases of irreducible characters for $R(S_3)$ and $R(S_4)$.

Problem 4 Let p be a prime. Recall that the automorphism group of the additive group $\mathbb{F}_p$ is the multiplicative group $(\mathbb{F}_p)^\times$, which is isomorphic to $\mathbb{Z}/(p-1)\mathbb{Z}$. The *affine group of* $\mathbb{F}_p$, denoted $\mathrm{Aff}(\mathbb{F}_p)$, is the semidirect product $\mathbb{F}_p \rtimes (\mathbb{F}_p)^\times$.

(a) (20 points) Let $\rho : \mathbb{F}_7 \to \mathbb{C}^\times$ be a one-dimensional faithful complex representation of the additive group $\mathbb{F}_7$. Find explicit matrices generating the image of the induced representation $\mathrm{Ind}_{\mathbb{F}_7}^{\mathrm{Aff}(\mathbb{F}_7)} \rho$.

(b) (30 points) Show that the induced representation $\mathrm{Ind}_{\mathbb{F}_p}^{\mathrm{Aff}(\mathbb{F}_p)} V$ is irreducible.

(c) (30 points) Describe the character table of $\mathrm{Aff}(\mathbb{F}_p)$.

Problem 5 (30 points) Exhibit two non-isomorphic G-sets whose corresponding permutation representations are isomorphic. (Hint: first determine the characters of $\mathrm{Ind}_H^{S_3} \sigma_H$ for all subgroups H of S_3, where σ_H denotes the trivial representation of H.)

Problem 6 Here we construct the *Burnside ring* $B(G)$ of a finite group G. This is the analog for G-sets what the representation ring is for G-representations.

Let $\mathcal{C}$ be the set of conjugacy classes of subgroups H in G. Let $B(G)$ be the free abelian group with basis $[G/H]$ where H varies over $\mathcal{C}$. Let $B^+(G)$ be the subset of $B(G)$ such that the coefficient of each basis element is non-negative.

(a) (20 points) Describe a canonical bijection from the set of isomorphism classes of all finite G-sets to the set $B^+(G)$.

(b) (20 points) Show that taking disjoint unions of G-sets corresponds to addition in $B(G)$.

(c) (20 points) Show that taking Cartesian products of G-sets gives rise to a multiplication on $B^+(G)$, which gives $B(G)$ a commutative unital ring structure.

(d) (20 points) Show that there is a canonical ring homomorphism $B(G) \to R(G)$ which takes $[G/H]$ to $\mathrm{Ind}_H^G(1_H)$ where 1_H is the trivial representation of H.