This assignment is "out of" 100 points, but there are far more than 100 points available. At the instructor's discretion some "overflow" above 100 may be counted towards your final grade at the end of the course, but you should not expect this.

You are **not** expected to write up a full solution to every problem, but you **are** expected to at least think about every problem. Writing up every single problem on the assignment is probably not a good use of your time.

Problem 1 Let k be an infinite field of characteristic 2. For $a, b \in k$, let $\rho_{a,b}$ be the representation of $(\mathbb{Z}/2\mathbb{Z})^2$ determined by

$$\rho_{a,b}(x, y) = \begin{pmatrix} 1 & ax + by \\ 0 & 1 \end{pmatrix}$$

(a) (20 points) Determine when $\rho_{a,b}$ is indecomposable.

(b) (30 points) Determine when $\rho_{a,b}$ is isomorphic to $\rho_{c,d}$.

(c) (10 points) Conclude that, when Maschke's theorem does not hold, there may be infinitely many isomorphism classes of indecomposable representations.

Problem 2 Let G be a finite group. For a finite-dimensional complex representation V of G, let χ_V denote the corresponding character.

(a) (30 points) Prove $\chi_{\mathrm{Sym}^2(V)}(g) = \chi_{\mathcal{S}^2(V)}(g) = \frac{1}{2}\left(\chi(g)^2 + \chi(g^2)\right)$ for $g \in G$.

(b) (10 points) Prove $\chi_{\mathrm{Alt}^2(V)}(g) = \chi_{\Lambda^2(V)}(g) = \frac{1}{2}\left(\chi(g)^2 - \chi(g^2)\right)$ for $g \in G$.

Problem 3 (30 points) Determine the character tables of the dihedral group D_8 of order 8 and the quaternionic group Q_8. Conclude that the entries of the character table do not determine the group.

Problem 4 (30 points) Recall that the character table of S_4 is as follows:

	e	(12)	$(12)(34)$	(123)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	2	-1	0
χ_4	3	1	-1	0	-1
χ_5	3	-1	-1	0	1

Compute the multiplication table for the ring $R(S_4)$ with respect to the basis of isomorphism classes of irreducible representations.

Problem 5 (30 points) Let G be a finite group and let k be a field of characteristic coprime to $|G|$. Suppose W is an irreducible representation of G over k and $V \cong W^{\oplus m}$ is isotypic. Show that $\mathrm{End}_k^G(V, V) \cong \mathrm{M}_m(k)$. (Hint: first show $W^{\oplus m} \cong k^m \otimes W$.)

Problem 6 Let G be a finite group. Let $G^\vee := \mathrm{Hom}_{group}(G, \mathbb{C}^\times)$ be the group of group homomorphisms from G to $\mathbb{C}^\times$.

(a) (10 points) Prove that G is abelian if and only if all its irreducible characters have degree 1.

(b) (10 points) Prove that the set of irreducible characters of G of degree 1 form a group isomorphic to $G^\vee$ under multiplication of characters.

(c) (10 points) Prove that if G is abelian then $(G^\vee)^\vee$ is canonically isomorphic to G. (Thus, when G is abelian, we call $G^\vee$ the *dual group* of G.)

(d) (10 points) Prove that $(G^\vee)^\vee$ is canonically isomorphic to the abelianization of G.

Problem 7 Let G be a finite group with center Z. Assume the base field is $\mathbb{C}$

(a) (10 points) Prove that if ρ is an irreducible representation, then $\rho(z)$ acts by scalar multiplication for all $z \in Z$.

(b) (10 points) Prove that, for all $g \in G$, we have $g \in Z$ if and only if $|\chi(g)| = \chi(1)$ for all irreducible characters χ.

(c) (10 points) Prove that all irreducible representations of G have degree $\leq \sqrt{|G/Z|}$. (Hint: $\sum_{g \in G} |\chi(g)|^2 = |G|$ for irreducible characters χ.)

(d) (10 points) Prove that if there exists a faithful irreducible representation, then Z is cyclic.

Problem 8 (30 points) Let G be a finite group of order n and let χ be an irreducible character of G. Show that the set

$$\{g \in G : |\chi(g)| = \chi(1)\}$$

is a normal subgroup of G.

Problem 9 (40 points) The group F_{20} is a semidirect product $\mathbb{Z}/5\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$ of cyclic groups. There is an irreducible 4-dimensional complex representation of F_{20} taking a pair of generators to the matrices

$$\begin{pmatrix} \zeta_5 & 0 & 0 & 0 \\ 0 & \zeta_5^2 & 0 & 0 \\ 0 & 0 & \zeta_5^4 & 0 \\ 0 & 0 & 0 & \zeta_5^3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

where ζ_5 is a primitive fifth root of unity. Determine the character table of F_{20}.