Problem 1 Let V and W be vector spaces over a field k. An element of $x \in V \otimes_k W$ is a *simple tensor* if there exist $v \in V$ and $w \in W$ such that $x = v \otimes w$. The *tensor rank* of $x \in V \otimes_k W$ is the minimal number n such that x can be written

$$x = \sum_{i=1}^{n} v_i \otimes w_i$$

for some $v_1, \ldots, v_n \in V$ and $w_1, \ldots, w_n \in W$. (By convention, 0 has tensor rank 0.)

(a) Let U be a complex vector space with basis $\{e_1, e_2\}$. Show that every element of $U \otimes_{\mathbb{C}} U$ has tensor rank 0, 1, or 2.

Solution: Follows immediately from the equation

$$ae_1 \otimes e_1 + be_1 \otimes e_2 + ce_2 \otimes e_1 + de_2 \otimes e_2 = e_1 \otimes (ae_1 + be_2) + e_2 \otimes (ce_1 + de_2)$$

that holds for general $a, b, c, d \in \mathbb{C}$.

(b) Recall that every element $x \in U \otimes_{\mathbb{C}} U$ can be written uniquely as

$$x = ae_1 \otimes e_1 + be_1 \otimes e_2 + ce_2 \otimes e_1 + de_2 \otimes e_2$$

for $a, b, c, d \in \mathbb{C}$. Determine a polynomial $f \in \mathbb{C}[y_1, y_2, y_3, y_4]$ such that x has tensor rank ≤ 1 if and only if $f(a, b, c, d) = 0$.

Solution: A general rank ≤ 1 tensor y can be written as

$$y = (se_1 + te_2) \otimes (ue_1 + ve_2)$$

for some $s, t, u, v \in \mathbb{C}$. Expanding, we find

$$y = sue_1 \otimes e_1 + sve_1 \otimes e_2 + tue_2 \otimes e_1 + tve_2 \otimes e_2.$$

Thus, the tensor x has the form y if and only if $(a, b, c, d) = (su, sv, tu, tv)$ for some $s, t, u, v \in \mathbb{C}$. The polynomial $f(y_1, y_2, y_3, y_4) = y_1y_4 - y_2y_3$ evaluates to zero if and only if this is the case.

(c) Let V, W be finite-dimensional k-vector spaces. Recall that $V^\vee \otimes_k W$ can be canonically identified with the set of linear transformations $V \to W$. Show that the tensor rank of $x \in V^\vee \otimes_k W$ is equal to the usual rank of the corresponding linear transformation $f : V \to W$.

Solution: By the usual process of row and column operations, one can put any matrix into a form where all entries are zero except for exactly $r = \text{rank}(f)$ ones on the diagonal. (Row operations alone gives reduced row echelon form, then column operations clear all non-pivot entries.) In other words, there exist choices of basis $v_1, \ldots, v_n$ for V and $w_1, \ldots, w_m$ for W such that

$$f(\bullet) = \sum_{i=1}^{r} v_i^\vee(\bullet) w_i$$

where $v_1^\vee, \ldots, v_n^\vee$ is the dual basis for $V^\vee$. This gives an upper bound of r on the tensor rank since

$$x = \sum_{i=1}^{r} v_i^\vee \otimes w_i$$

under the correspondence. Conversely, if x has tensor rank t, then

$$x = \sum_{i=1}^{t} v_i^\vee \otimes w_i$$

for some $v_1^\vee, \ldots, v_t^\vee \in V^\vee$ and $w_1, \ldots, w_r \in W$. The image of f is a subset of $\operatorname{span}_k\{w_1, \ldots, w_r\}$, thus $\operatorname{rank}(f) \leq t$.

Problem 2 Prove that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \oplus \mathbb{C}$ are isomorphic as $\mathbb{R}$-algebras.

Problem 3 Suppose $f : V \to V$ is a linear transformation of an n-dimensional vector space. For all non-negative d we have a map

$$\Lambda^d(f) : \Lambda^d(V) \to \Lambda^d(V)$$

given by

$$v_1 \wedge \cdots \wedge v_d \mapsto f(v_1) \wedge \cdots \wedge f(v_d)$$

for $v_1, \ldots, v_d \in V$ and extended by linearity.

(a) Since $\Lambda^n(V)$ is 1-dimensional, $\Lambda^n(f)$ is just multiplication by a scalar. Prove that $\Lambda^n(f)$ is multiplication by $\det(f)$.

(b) Prove that

$$e_i = \operatorname{tr}(\Lambda^i(f))$$

for all $1 \leq i \leq n$, where

$$\chi_f(t) = \sum_{i=0}^{n} (-1)^i e_i t^{n-i}$$

is the characteristic polynomial.

Problem 4 Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^3$ and let $\vec{i}$, $\vec{j}$, $\vec{k}$ be the standard basis. Show that

$$\vec{u} \wedge \vec{v} = w_1(\vec{j} \wedge \vec{k}) + w_2(\vec{k} \wedge \vec{i}) + w_3(\vec{i} \wedge \vec{j})$$

where $\vec{w} = (w_1, w_2, w_3)$ is the cross product $\vec{w} = \vec{u} \times \vec{v}$.

Problem 5 In the following, n is a non-negative integer. Determine all irreducible subrepresentations of the following linear representations.

(a) The two-dimensional real representation ρ of $\mathbb{Z}/n\mathbb{Z}$ where $\rho(1)$ acts by rotation about the origin $2\pi/n$ radians counterclockwise.

(b) The two-dimensional complex representation ρ of $\mathbb{Z}/n\mathbb{Z}$ where $\rho(1)$ acts by rotation about the origin $2\pi/n$ radians counterclockwise.

(c) The two-dimensional complex representation ρ of the dihedral group $D_{2n} = \langle s, r | s^2, r^n, srsr \rangle$ where $\rho(s)$ acts by reflection about the x-axis and $\rho(r)$ acts by rotation about the origin $2\pi/n$ radians counterclockwise.

(d) The complex regular representation of the cyclic group $\mathbb{Z}/n\mathbb{Z}$.

(e) The regular representation of the cyclic group $\mathbb{Z}/p\mathbb{Z}$ over the field $\mathbb{F}_p$ where p is a prime.

Problem 6 Let (V, ρ) be a finite-dimensional linear representation of a group G over a field k. Define a set map $\det(\rho) : G \to \mathrm{GL}_1(k)$ via $\det(\rho)(g) := \det(\rho(g))$ for $g \in G$.

(a) Prove that $\det(\rho)$ is a linear representation.

(b) Exhibit a representation (V, ρ) such that $\det(\rho)$ is non-trivial and G is a non-abelian group.

(c) Prove that if G is a nonabelian simple group, then $\det(\rho)$ is trivial.

Problem 7 Let k be an arbitrary field, V be a k-vector space of dimension n, and d be a non-negative integer.

(a) Show that there is a unique invertible linear transformation

$$\Phi : \Lambda^d(V^\vee) \cong \left(\Lambda^d(V)\right)^\vee$$

such that

$$\Phi(f_1 \wedge \cdots \wedge f_d)(v_1 \wedge \cdots \wedge v_d) = \det \begin{pmatrix} f_1(v_1) & \cdots & f_d(v_1) \\ \vdots & \ddots & \vdots \\ f_1(v_d) & \cdots & f_d(v_d) \end{pmatrix}$$

for all $v_1, \ldots, v_d \in V$ and $f_1, \ldots, f_d \in V^\vee$.

(b) Let $e_1, \ldots, e_n$ be a basis for V and let $x_1, \ldots, x_n$ be the dual basis. Prove that the "obvious" isomorphism

$$\Lambda^d(V^\vee) \to \left(\Lambda^d(V)\right)^\vee$$

defined by

$$(x_1 \wedge \cdots \wedge x_n) \mapsto (e_1 \wedge \cdots \wedge e_n)^\vee$$

agrees with Φ and thus does not depend on choice of basis.

Problem 8 Let k be an arbitrary field, V be a k-vector space of dimension n, and d be a non-negative integer. Let $e_1, \ldots, e_n$ be a basis for V and let $x_1, \ldots, x_n$ be the dual basis.

(a) There is an "obvious" isomorphism

$$\Psi : \mathcal{S}^d(V^\vee) \to \left(\mathcal{S}^d(V)\right)^\vee$$

defined by

$$\Psi\left(x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}\right) = \left(e_1^{a_1} e_2^{a_2} \cdots e_n^{a_n}\right)^\vee$$

for all $0 \leq a_1, \ldots, a_n \leq d$ such that $a_1 + \ldots + a_n = d$. Show that Ψ is not canonical by demonstrating that it depends on the choice of basis.

(b) Define

$$\Phi : \mathcal{S}^d(V^\vee) \to \left(\mathcal{S}^d(V)\right)^\vee$$

via

$$\Phi\left(x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}\right) = \frac{d!}{a_1! \cdots a_n!} \left(e_1^{a_1} e_2^{a_2} \cdots e_n^{a_n}\right)^\vee$$

for all $0 \leq a_1, \ldots, a_n \leq d$ such that $a_1 + \ldots + a_n = d$. Prove that, despite its definition, Φ does not depend on the choice of basis.

(c) Show that Φ is an isomorphism if and only if $\mathrm{char}(k) > d$.