This assignment is "out of" 100 points, but there are far more than 100 points available. At the instructor's discretion some "overflow" above 100 may be counted towards your final grade at the end of the course, but you should not expect this.

You are **not** expected to write up a full solution to every problem, but you **are** expected to at least think about every problem. Writing up every single problem on the assignment is probably not a good use of your time.

Errata (2023/01/25): Clarified wording for 1(b).

Errata (2023/02/02): Added nonabelian hypothesis to 6(c).

Problem 1 Let V and W be vector spaces over a field k. An element of $x \in V \otimes_k W$ is a *simple tensor* if there exist $v \in V$ and $w \in W$ such that $x = v \otimes w$. The *tensor rank* of $x \in V \otimes_k W$ is the minimal number n such that x can be written

$$x = \sum_{i=1}^{n} v_i \otimes w_i$$

for some $v_1, \dots, v_n \in V$ and $w_1, \dots, w_n \in W$. (By convention, 0 has tensor rank 0.)

(a) (20 points) Let U be a complex vector space with basis $\{e_1, e_2\}$. Show that every element of $U \otimes_{\mathbb{C}} U$ has tensor rank 0, 1, or 2.

(b) (20 points) Recall that every element $x \in U \otimes_{\mathbb{C}} U$ can be written uniquely as

$$x = ae_1 \otimes e_1 + be_1 \otimes e_2 + ce_2 \otimes e_1 + de_2 \otimes e_2$$

for $a, b, c, d \in \mathbb{C}$. Determine a polynomial $f \in \mathbb{C}[y_1, y_2, y_3, y_4]$ such that x has tensor rank ≤ 1 if and only if $f(a, b, c, d) = 0$.

(c) (20 points) Let V, W be finite-dimensional k-vector spaces. Recall that $V^\vee \otimes_k W$ can be canonically identified with the set of linear transformations $V \to W$. Show that the tensor rank of $x \in V^\vee \otimes_k W$ is equal to the usual rank of the corresponding linear transformation $f : V \to W$.

Problem 2 (40 points) Prove that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \oplus \mathbb{C}$ are isomorphic as $\mathbb{R}$-algebras.

Problem 3 Suppose $f : V \to V$ is a linear transformation of an n-dimensional vector space. For all non-negative d we have a map

$$\Lambda^d(f) : \Lambda^d(V) \to \Lambda^d(V)$$

given by

$$v_1 \wedge \cdots \wedge v_d \mapsto f(v_1) \wedge \cdots \wedge f(v_d)$$

for $v_1, \dots, v_d \in V$ and extended by linearity.

(a) (20 points) Since $\Lambda^n(V)$ is 1-dimensional, $\Lambda^n(f)$ is just multiplication by a scalar. Prove that $\Lambda^n(f)$ is multiplication by $\det(f)$.

(b) (60 points) Prove that

$$e_i = \mathrm{tr}(\Lambda^i(f))$$

for all $1 \leq i \leq n$, where

$$\chi_f(t) = \sum_{i=0}^{n} (-1)^i e_i t^{n-i}$$

is the characteristic polynomial.

Problem 4 (30 points) Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^3$ and let $\vec{i}$, $\vec{j}$, $\vec{k}$ be the standard basis. Show that

$$\vec{u} \wedge \vec{v} = w_1(\vec{j} \wedge \vec{k}) + w_2(\vec{k} \wedge \vec{i}) + w_3(\vec{i} \wedge \vec{j})$$

where $\vec{w} = (w_1, w_2, w_3)$ is the cross product $\vec{w} = \vec{u} \times \vec{v}$.

Problem 5 In the following, n is a non-negative integer. Determine all irreducible subrepresentations of the following linear representations.

(a) (10 points) The two-dimensional real representation ρ of $\mathbb{Z}/n\mathbb{Z}$ where $\rho(1)$ acts by rotation about the origin $2\pi/n$ radians counterclockwise.

(b) (10 points) The two-dimensional complex representation ρ of $\mathbb{Z}/n\mathbb{Z}$ where $\rho(1)$ acts by rotation about the origin $2\pi/n$ radians counterclockwise.

(c) (10 points) The two-dimensional complex representation ρ of the dihedral group $D_{2n} = \langle s, r | s^2, r^n, srsr \rangle$ where $\rho(s)$ acts by reflection about the x-axis and $\rho(r)$ acts by rotation about the origin $2\pi/n$ radians counterclockwise.

(d) (10 points) The complex regular representation of the cyclic group $\mathbb{Z}/n\mathbb{Z}$.

(e) (10 points) The regular representation of the cyclic group $\mathbb{Z}/p\mathbb{Z}$ over the field $\mathbb{F}_p$ where p is a prime.

Problem 6 Let (V, ρ) be a finite-dimensional linear representation of a group G over a field k. Define a set map $\det(\rho) : G \to \mathrm{GL}_1(k)$ via $\det(\rho)(g) := \det(\rho(g))$ for $g \in G$.

(a) (20 points) Prove that $\det(\rho)$ is a linear representation.

(b) (20 points) Exhibit a representation (V, ρ) such that $\det(\rho)$ is non-trivial and G is a non-abelian group.

(c) (20 points) Prove that if G is a nonabelian simple group, then $\det(\rho)$ is trivial.

Problem 7 Let k be an arbitrary field, V be a k-vector space of dimension n, and d be a non-negative integer.

(a) (30 points) Show that there is a unique invertible linear transformation

$$\Phi : \Lambda^d(V^\vee) \cong \left(\Lambda^d(V)\right)^\vee$$

such that

$$\Phi(f_1 \wedge \cdots \wedge f_d)(v_1 \wedge \cdots \wedge v_d) = \det \begin{pmatrix} f_1(v_1) & \cdots & f_d(v_1) \\ \vdots & \ddots & \vdots \\ f_1(v_d) & \cdots & f_d(v_d) \end{pmatrix}$$

for all $v_1, \ldots, v_d \in V$ and $f_1, \ldots, f_d \in V^\vee$.

(b) (20 points) Let $e_1, \ldots, e_n$ be a basis for V and let $x_1, \ldots, x_n$ be the dual basis. Prove that the "obvious" isomorphism

$$\Lambda^d(V^\vee) \to \left(\Lambda^d(V)\right)^\vee$$

defined by

$$(x_1 \wedge \cdots \wedge x_n) \mapsto (e_1 \wedge \cdots \wedge e_n)^\vee$$

agrees with Φ and thus does not depend on choice of basis.

Problem 8 Let k be an arbitrary field, V be a k-vector space of dimension n, and d be a non-negative integer. Let $e_1, \ldots, e_n$ be a basis for V and let $x_1, \ldots, x_n$ be the dual basis.

(a) (20 points) There is an "obvious" isomorphism

$$\Psi : \mathcal{S}^d(V^\vee) \to \big(\mathcal{S}^d(V)\big)^\vee$$

defined by

$$\Psi\left(x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}\right) = \left(e_1^{a_1} e_2^{a_2} \cdots e_n^{a_n}\right)^\vee$$

for all $0 \leq a_1, \ldots, a_n \leq d$ such that $a_1 + \ldots + a_n = d$. Show that Ψ is not canonical by demonstrating that it depends on the choice of basis.

(b) (60 points) Define

$$\Phi : \mathcal{S}^d(V^\vee) \to \big(\mathcal{S}^d(V)\big)^\vee$$

via

$$\Phi\left(x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}\right) = \frac{d!}{a_1! \cdots a_n!} \left(e_1^{a_1} e_2^{a_2} \cdots e_n^{a_n}\right)^\vee$$

for all $0 \leq a_1, \ldots, a_n \leq d$ such that $a_1 + \ldots + a_n = d$. Prove that, despite its definition, Φ does not depend on the choice of basis.

(c) (20 points) Show that Φ is an isomorphism if and only if $\mathrm{char}(k) > d$.