Problem 1. Find the maximum and minimum values of $f(x, y) = x^2 + y^2$ subject to the constraint $x^2 + y^2 = 2x + 4y$.

Solution: Rewriting the constraint as $g(x, y) = x^2 + y^2 - 2x - 4y$, we use the technique of Lagrange multipliers. We want to solve the system

$$\begin{aligned} 2x &= \lambda(2x - 2) \\ 2y &= \lambda(2y - 4) \\ 0 &= x^2 + y^2 - 2x - 4y. \end{aligned}$$

Observe that $x = 1$ forces $2 = 0$ in the first equation and $y = 2$ forces $4 = 0$ in the second equation; both of which are contradictions. Thus, we may safely divide by $x-1$ and $y-2$. Solving the first two equations for λ, we obtain the equation

$$\frac{x}{x-1} = \frac{y}{y-2},$$

which simplifies to $2x = y$. Substituting this into the last equation we obtain $0 = 5x^2 - 10x$. Thus $(x, y) = (0, 0)$ or $(x, y) = (2, 4)$. We have $f(0, 0) = 0$ as our minimum value and $f(2, 4) = 20$ as our maximum value.

Problem 2. Find $\int_0^1 \int_0^2 \int_0^3 x^2y + z \; dy \; dx \; dz$.

Solution:

$$\begin{aligned} &\int_0^1 \int_0^2 \int_0^3 x^2y + z \; dy \; dx \; dz \\ &= \int_0^1 \int_0^2 \frac{1}{2}x^2y^2 + yz\Big|_{y=0}^{y=3} \; dx \; dz \\ &= \int_0^1 \int_0^2 \frac{9}{2}x^2 + 3z \; dx \; dz \\ &= \int_0^1 \frac{3}{2}x^3 + 3xz\Big|_{x=0}^{x=2} \; dz \\ &= \int_0^1 12 + 6z \; dz \\ &= 12z + 3z^2\Big|_{z=0}^{z=1} = 15 \end{aligned}$$

Problem 3. Find $\int_0^1 \int_y^{y^2} x + 2 \; dx \; dy$.

Solution:

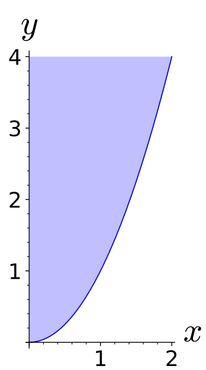
$$\begin{aligned}
&\int_0^1 \int_y^{y^2} x + 2 \; dx \; dy \\
&= \int_0^1 \frac{1}{2}x^2 + 2x \Big|_{x=y}^{x=y^2} \; dy \\
&= \int_0^1 \frac{1}{2}y^4 + \frac{3}{2}y^2 - 2y \; dy \\
&= \frac{1}{10}y^5 + \frac{1}{2}y^3 - y^2 \Big|_{y=0}^{y=1} \\
&= -\frac{2}{5}
\end{aligned}$$

Problem 4. Change the order of integration in

$$\int_0^2 \int_{x^2}^4 f(x,y) \; dy \; dx.$$

(This may involve breaking the integral up into multiple integrals.)

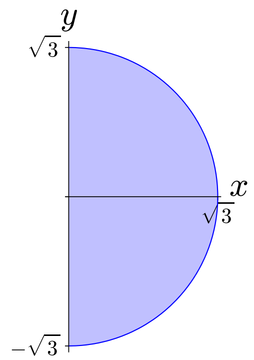
Solution:



The new integral is: $\int_0^4 \int_0^{\sqrt{y}} f(x,y) \; dx \; dy$

Problem 5. Find $\iint_R x\ dA$ where R is the semicircular region given by the inequalities $x^2 + y^2 \leq 3$ and $x \geq 0$.

Solution:



Using polar coordinates, the region is given by $r \leq \sqrt{3}$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Since $x = r\cos(\theta)$, we have

$$\begin{aligned}\iint_R x\ dA &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\sqrt{3}} (r\cos(\theta))r\ dr\ d\theta \\ &= \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\theta)\ d\theta\right)\left(\int_0^{\sqrt{3}} r^2\ dr\right) \\ &= \left(\sin(\theta)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right)\left(\frac{1}{3}r^3\Big|_0^{\sqrt{3}}\right) = 2\sqrt{3}.\end{aligned}$$

Alternatively: The integral can be set up in Cartesian coordinates as either

$$\iint_R x\ dA = \int_{-\sqrt{3}}^{\sqrt{3}} \int_0^{\sqrt{3-y^2}} x\ dx\ dy$$

or

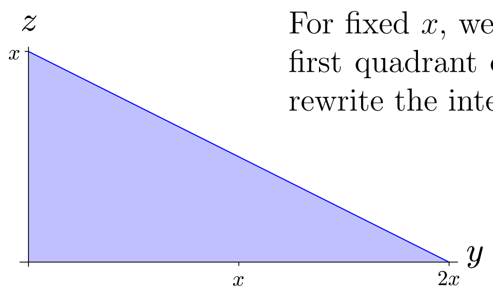
$$\iint_R x\ dA = \int_0^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} x\ dy\ dx.$$

Problem 6. Change the order of integration of y and z in

$$\int_1^2 \int_0^x \int_0^{2x-2z} f(x,y,z)\ dy\ dz\ dx$$

(This may involve breaking the integral up into multiple integrals.)

Solution:



For fixed x, we are integrating the region to the right of the line $y = 2x - 2z$ in the first quadrant of the yz-plane. Rewriting the equation of the line as $z = x - \frac{y}{2}$ we rewrite the integral as

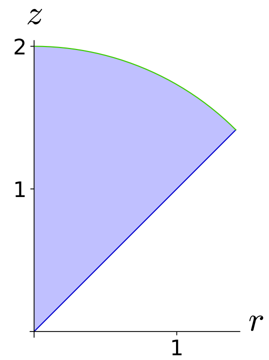
$$\int_1^2 \int_0^{2x} \int_0^{x-\frac{y}{2}} f(x,y,z)\ dz\ dy\ dx$$

Problem 7. Rewrite the following integral using spherical coordinates

$$\int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} r\ dz\ dr\ d\theta.$$

(Do not evaluate the integral.)

Solution:

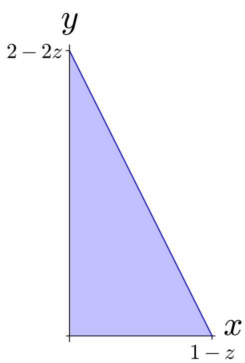


We see that the region is bounded below by the cone $\varphi = \pi/4$ and bounded above by the sphere $\rho = 2$. Since $dV = rdrd\theta dz = \rho^2 \sin(\varphi) d\rho d\varphi d\theta$, the integral can be rewritten

$$\int_0^{2\pi} \int_0^{\pi/4} \int_0^{2} \rho^2 \sin(\varphi)\ d\rho\ d\varphi\ d\theta.$$

Problem 8. Find the volume of the tetrahedron in the first octant bounded by the coordinate planes and the plane passing through the points $(1, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 1)$.

Solution:



The plane passing through the three given points is

$$2x + y + 2z = 2,$$

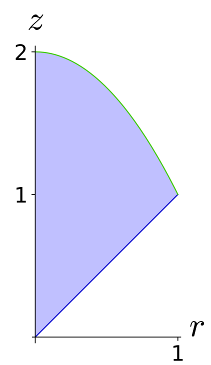
which can be determined by inspection. (Alternatively, if the points are P, Q, R, then the normal vector can be determined by computing $\overrightarrow{PQ} \times \overrightarrow{PR}$ and then plugging in a point to obtain the constant.)

To find the volume, we will integrate the constant function 1 on the tetrahedron. Thus

$$\begin{aligned} V &= \int_0^1 \int_0^{2-2z} \int_0^{1-\frac{y}{2}-z} 1 \, dx \, dy \, dz \\ &= \int_0^1 \int_0^{2-2z} 1 - \frac{y}{2} - z \, dy \, dz \\ &= \int_0^1 y - \frac{y^2}{4} - yz \Big|_{y=0}^{y=2-2z} dz \\ &= \int_0^1 (2-2z) - \frac{(2-2z)^2}{4} - (2-2z)z \, dz \\ &= \int_0^1 z^2 - 2z + 1 \, dz \\ &= \frac{1}{3}z^3 - z^2 + z \Big|_0^1 = \frac{1}{3} \end{aligned}$$

Problem 9. Find $\iiint_E z\, dV$ where E is the solid region below by the cone $z = \sqrt{x^2+y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$

Solution:



In cylindrical coordinates, $z = r$ is the cone and $z = 2 - r^2$ is the paraboloid. We need to determine where they intersect. This amounts to solving $r = z = 2 - r^2$, which gives the quadratic equation $r^2 + r - 2 = 0$. The roots are $r = -2$ and $r = 1$. Since r is positive, we conclude that the two surfaces meet when $r = 1$.

We can now set up the integral as follows:

$$\begin{aligned}
\iiint_E z\, dV &= \int_0^{2\pi}\int_0^1\int_r^{2-r^2} (z) r\; dz\; dr\; d\theta \\
&= \left(\int_0^{2\pi} 1\; d\theta\right)\left(\int_0^1 \frac{z^2}{2} r\Big|_r^{2-r^2}\; dr\right) \\
&= 2\pi\left(\int_0^1 2r - \frac{5}{2}r^3 + \frac{1}{2}r^5\; dr\right) \\
&= 2\pi\left(r^2 - \frac{5}{8}r^4 + \frac{1}{12}r^6\Big|_0^1\right) \\
&= 2\pi\left(1 - \frac{5}{8} + \frac{1}{12}\right) = \frac{11}{12}\pi
\end{aligned}$$

Alternatively: The integral can be set up in Cartesian coordinates as

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{2-x^2-y^2} z\; dz\; dy\; dx,$$

which is a little unpleasant. In spherical coordinates it is

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{f(\varphi)} \rho^3\cos(\varphi)\sin(\varphi)\; d\rho\; d\varphi\; d\theta,$$

where $f(\varphi)$ is an unpleasant function involving both radicals and trigonometric functions. It is not practical to solve this problem using spherical coordinates.