Problem 1. Describe the domain and range of the function

$$f(x,y) = \frac{1}{x^2+y^2}.$$

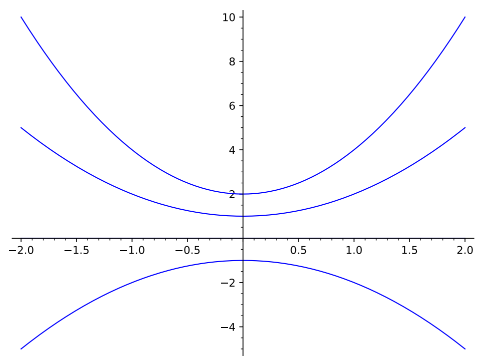
Solution: Observe that $x^2 + y^2 \geq 0$ for all $(x,y) \in \mathbb{R}^2$; equality holds if and only if $(x,y) = (0,0)$. Thus f is defined unless $(x,y) = (0,0)$. Moreover, we have $f(1/\sqrt{z}, 0) = z$ for all $z > 0$. Thus the domain of f is all of $\mathbb{R}^2$ except $(0,0)$, while the range of f is the set of positive real numbers. (Note: interval notation does not make sense for the domain since it is 2-dimensional.)

Problem 2. Find and sketch the level curves $f(x,y) = c$ on the same set of axes for the function

$$f(x,y) = \frac{y}{x^2+1}$$

at the values $c = -1, 0, 1, 2$.

Solution: Rearranging $f(x,y) = c$, we have $y = c(x^2+1)$. This yields $y = -x^2 - 1$ for $c = -1$, $y = 0$ for $c = 0$, $y = x^2 + 1$ for $c = 1$, and $y = 2x^2 + 2$ for $c = 2$.



Problem 3. Find the limit $\displaystyle\lim_{(x,y)\to(6,2)} \frac{e^{3y-x}}{x^2-y}$.

Solution:

$$\lim_{(x,y)\to(6,2)} \frac{e^{3y-x}}{x^2-y} = \frac{e^{3(2)-6}}{6^2-2} = \frac{1}{34}$$

Problem 4. Find the limit $\displaystyle\lim_{(x,y)\to(2,2)} \frac{x^4-y^4}{x^2-y^2}$.

Solution:

$$\lim_{(x,y)\to(2,2)} \frac{x^4-y^4}{x^2-y^2} = \lim_{(x,y)\to(2,2)} \frac{(x^2+y^2)(x^2-y^2)}{x^2-y^2} = \lim_{(x,y)\to(2,2)} (x^2+y^2) = 8$$

Problem 5. Show the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy-y^3}{x^2+y^2}$ does not exist by considering the paths $y = -x$ and $y = x$.

Solution: For $y = -x$, we find

$$\lim_{x\to 0} \frac{x(-x) - (-x)^3}{x^2 + (-x)^2} = \lim_{x\to 0} \frac{-x^2 + x^3}{2x^2} = \lim_{x\to 0} \frac{-1 + x}{2} = -\frac{1}{2},$$

but for $y = x$, we find

$$\lim_{x\to 0} \frac{x(x) - (x)^3}{x^2 + (x)^2} = \lim_{x\to 0} \frac{x^2 - x^3}{2x^2} = \lim_{x\to 0} \frac{1 - x}{2} = \frac{1}{2}.$$

Since the limits disagree along distinct paths, the original limit does not exist.

Problem 6. If $z = x^2 - 2xy + 3y$, $x = u^2 + v$, and $y = 2uv + 1$, find $\frac{\partial z}{\partial u}$.

Solution: First, we compute some useful intermediate quantities:

$$\frac{\partial z}{\partial x} = 2x - 2y, \quad \frac{\partial z}{\partial y} = -2x + 3, \quad \frac{\partial x}{\partial u} = 2u, \quad \frac{\partial y}{\partial u} = 2v.$$

Now, by chain rule:

$$\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\
&= (2x - 2y)(2u) + (-2x + 3)(2v) \\
&= (2(u^2 + v) - 2(2uv + 1))(2u) + (-2(u^2 + v) + 3)(2v) \\
&= 4u^3 + 4uv - 8u^2v - 4u - 4u^2v - 4v^2 + 6v \\
&= 4u^3 - 12u^2v + 4uv - 4v^2 - 4u + 6v
\end{aligned}$$

Problem 7. Find the equation for the tangent plane to the surface

$$z = 3x^2 + 4y^2 - 2xy + 3$$

at $(x, y) = (-1, 1)$.

Solution: With $z = f(x, y) = 3x^2 + 4y^2 - 2xy + 3$, we find

$$\frac{\partial f}{\partial x} = 6x - 2y \text{ and } \frac{\partial f}{\partial y} = 8y - 2x.$$

The equation of the tangent plane is

$$\begin{aligned}
z &= f(-1, 1) + f_x(-1, 1)(x + 1) + f_x(-1, 1)(y - 1) \\
&= 12 - 8(x + 1) + 10(y - 1) \\
&= -6 - 8x + 10y.
\end{aligned}$$

Problem 8. Given $f(x, y) = \sqrt{x^2 + y^2}$, approximate $f(4.1, 2.9)$ using a linear approximation at the point $(4, 3)$.

Solution: We find

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \text{ and } \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}.$$

Thus $f(4,3) = 5$, $f_x(4,3) = \frac{4}{5}$, and $f_y(4,3) = \frac{3}{5}$. The linear approximation is

$$\begin{aligned} f(4.1, 2.9) &\approx f(4,3) + f_x(4,3)(4.1-4) + f_y(4,3)(2.9-3) \\ &= 5 + (0.8)(0.1) + (0.6)(-0.1) = 5.02. \end{aligned}$$

Problem 9. Find the directional derivative of the function

$$f(x,y) = e^{x^2-y^2}$$

at the point $(1,2)$ in the direction of $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$.

Solution: We find

$$\frac{\partial f}{\partial x} = 2xe^{x^2-y^2} \quad \frac{\partial f}{\partial y} = -2ye^{x^2-y^2}$$

and

$$f_x(1,2) = 2e^{-3} \quad f_x(1,2) = -4e^{-3}.$$

Observe that $\mathbf{u}$ is already a unit vector, thus

$$(D_{\mathbf{u}}f)(1,2) = \mathbf{u} \cdot \nabla f(1,2) = \frac{1}{\sqrt{2}}(2e^{-3}) + \frac{1}{\sqrt{2}}(-4e^{-3}) = -\sqrt{2}e^{-3}$$

Problem 10. For the function

$$f(x,y) = x^3 - y^3 - 2xy + 7,$$

find all the critical points and use the second derivative test to determine, if possible, whether each is a maximum, minimum, or saddle point.

Solution: The range of the function is all real numbers, so there cannot be any global minima or maxima.

First, we find all critical points. We need to solve the simultaneous system

$$\begin{aligned} 0 &= f_x = 3x^2 - 2y \\ 0 &= f_y = -3y^2 - 2x. \end{aligned}$$

From the first equation, we have $y = \frac{3}{2}x^2$. Substituting this into the second equation we obtain

$$0 = -3\left(\frac{3}{2}x^2\right)^2 - 2x = \left(\frac{-3 \cdot 3^2}{2^2}\right)x^4 - 2x = -2x\left(\frac{3^3}{2^3}x^3 + 1\right).$$

Thus, either $x = 0$ or $\left(\frac{3}{2}x\right)^3 + 1 = 0$; the second equation only has the one real solution $x = -\frac{2}{3}$. Thus, the critical points are $(0,0)$ and $(-\frac{2}{3}, \frac{2}{3})$.

We compute the Jacobian determinant

$$D = \det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \det \begin{pmatrix} 6x & -2 \\ -2 & -6y \end{pmatrix} = -36xy - 4.$$

Since $D(0,0) = -4$ is negative, the point $(0,0)$ is a saddle point. Since $D(-\frac{2}{3}, \frac{2}{3}) = 12$ is positive and $f_{xx}(-\frac{2}{3}, \frac{2}{3}) - 12$ is negative, the point $(\frac{2}{3}, \frac{2}{3})$ is a local maximum.

Problem 11. Find the absolute extrema of the function

$$f(x,y) = 6xy - 4x^3 - 3y^2$$

on the closed and bounded rectangle given by $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$.

Solution: First, we find critical points by solving the system

$$\begin{aligned} 0 &= f_x = 6y - 12x^2 \\ 0 &= f_y = 6x - 6y. \end{aligned}$$

The second equation gives $x = y$. Substituting this into the first equation gives $6y - 12y^2 = 0$ which has solutions $y = 0$ and $y = \frac{1}{2}$. Thus the critical points are $(0,0)$ and $\left(\frac{1}{2}, \frac{1}{2}\right)$.

Now we consider the boundary. First, let us take the line where $x = -1$. Then we want to optimize

$$g(y) = f(-1, y) = -6y + 4 - 3y^2.$$

We solve $g'(y) = -6 - 6y = 0$ as $y = -1$. Thus $(-1,-1)$ is another candidate.

For $x = 1$, we optimize

$$h(y) = f(1, y) = 6y - 4 - 3y^2.$$

We find $h'(y) = 6 - 6y = 0$; so $y = 1$. Thus $(1,1)$ is another candidate.

For $y = -1$, we optimize

$$j(x) = f(x, -1) = -6x - 4x^3 - 3.$$

We find $j'(x) = -6 - 12x^2 = 0$; which has no real solutions.

For $y = 1$, we optimize

$$j(x) = f(x, 1) = 6x - 4x^3 - 3.$$

We find $j'(x) = 6 - 12x^2 = 0$; which has solution $x = \pm\frac{1}{\sqrt{2}}$. Thus $\left(\frac{1}{\sqrt{2}}, 1\right)$ and $\left(-\frac{1}{\sqrt{2}}, 1\right)$ are also candidates.

We also need all the four corners $(\pm 1, \pm 1)$, which are the endpoints.

Now we compare all the candidates:

$$\begin{aligned} f(-1,-1) &= 7 & f(1,1) &= -1 \\ f(-1,1) &= -5 & f(1,-1) &= -13 \\ f(0,0) &= 0 & f\left(\frac{1}{2}, \frac{1}{2}\right) &= \frac{1}{4} \\ f\left(\frac{1}{\sqrt{2}}, 1\right) &= -2\sqrt{2} - 3 & f\left(-\frac{1}{\sqrt{2}}, 1\right) &= 2\sqrt{2} - 3 \end{aligned}$$

We find a global maximum of 7 at $(-1,-1)$ and a global minimum of -13 at $(1,-1)$.
