Problem 1. Given vectors $\mathbf{a} = \langle 2, 5, -1\rangle$ and $\mathbf{b} = \langle 2, -1, 1\rangle$, express the vector $5\mathbf{a} - \mathbf{b}$ in component form.

Solution: $5\mathbf{a} - \mathbf{b} = \langle 10, 25, -5\rangle - \langle 2, -1, 1\rangle = \langle 8, 26, -6\rangle$.

Problem 2. A vector $\mathbf{v}$ has initial point $(1, 2, 4)$ and terminal point $(0, 2, 2)$. Find the unit vector in the direction of $\mathbf{v}$. Express the answer in component form.

Solution: First, we find $\mathbf{v} = (0, 2, 2) - (1, 2, 4) = \langle -1, 0, -2\rangle$. Then $\|\mathbf{v}\| = \sqrt{(-1)^2 + (0)^2 + (-2)^2} = \sqrt{5}$. The unit vector is thus $\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \left\langle \dfrac{-1}{\sqrt{5}}, 0, \dfrac{-2}{\sqrt{5}} \right\rangle$.

Problem 3. Find the component form of the two-dimensional vector $\mathbf{u}$ where $\|\mathbf{u}\| = 3$ and the angle the vector makes with the positive direction of the x-axis is $30°$ in a counter-clockwise direction.

Solution: $\mathbf{u} = \langle \|\mathbf{u}\| \cos(\theta), \|\mathbf{u}\| \sin(\theta)\rangle = \langle 3\cos(30°), 3\sin(30°)\rangle = \left\langle \dfrac{3\sqrt{3}}{2}, \dfrac{3}{2} \right\rangle$.

Problem 4. Given vectors $\mathbf{a} = \mathbf{i} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j}$, and $\mathbf{c} = \mathbf{j} + \mathbf{k}$, compute $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.

Solution: $(\mathbf{a} \cdot \mathbf{b})\mathbf{c} = (\langle 1, 0, 1\rangle \cdot \langle 2, -1, 0\rangle)\langle 0, 1, 1\rangle = 2\langle 0, 1, 1\rangle = \langle 0, 2, 2\rangle$.

Problem 5. Given vectors $\mathbf{a} = \langle 1, 2, 3\rangle$ and $\mathbf{b} = \langle 0, 2, 1\rangle$, compute $\mathbf{a} \times \mathbf{b}$.

Solution: $\mathbf{a} \times \mathbf{b} = \langle -4, -1, 2\rangle$.

Problem 6. Find a vector $\mathbf{c}$ that is orthogonal to both $\mathbf{a} = \langle 1, 1, 1\rangle$ and $\mathbf{b} = \langle 1, 2, 0\rangle$.

Solution: One way to construct such a $\mathbf{c}$ is to use the cross product. Thus $\mathbf{c} = \mathbf{a} \times \mathbf{b} = \langle -2, 1, 1\rangle$.

Problem 7. Determine the vector projection $\text{proj}_{\mathbf{u}}\, \mathbf{v}$ of the vector $\mathbf{v} = \langle 2, 1, 1\rangle$ onto the vector $\mathbf{u} = \langle 1, 1, 1\rangle$.

Solution: $\text{proj}_{\mathbf{u}}\, \mathbf{v} = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u} = \dfrac{4}{3}\langle 1, 1, 1\rangle = \left\langle \dfrac{4}{3}, \dfrac{4}{3}, \dfrac{4}{3} \right\rangle$

Problem 8. Find a normal vector $\mathbf{n}$ to the plane with equation $x - 2y + 3z - 60 = 0$.

Solution: $\mathbf{n} = \langle 1, -2, 3\rangle$

Problem 9. Let L be the line passing through the point $P = (0, 1, 1)$ with direction $\mathbf{v} = \langle 3, 2, 1\rangle$. Find symmetric equations of the line L.

Solution: $\dfrac{x}{3} = \dfrac{y-1}{2} = \dfrac{z-1}{1}$.

Problem 10. Find the general equation of the plane passing through $P = (0, 0, 1)$, $Q = (2, 3, 0)$, and $R = (1, 0, -1)$.

Solution: We find a normal vector $\mathbf{n}$ to the plane via the cross product

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \langle 2, 3, -1\rangle \times \langle 1, 0, 2\rangle = \langle -6, 3, -3\rangle.$$

Now our plane satisfies $-6x + 3y - 3z = d$ for some d. Since P lies on the plane $-6(0) + 3(0) - 3(1) = d$. Thus $d = -3$. Thus the equation of our plane is $-6x + 3y - 3z = -3$ or $2x - y + z = 1$.

Problem 11. Find the distance from the point $P = (4, 1, 1)$ to the plane with equation $2x - y + z = 0$.

Solution: A normal vector to the plane is $\mathbf{n} = \langle 2, -1, 1\rangle$. Note that $Q = (0, 0, 0)$ lies on the plane. Thus, we want to compute the scalar component of the projection of $\overrightarrow{QP} = \langle 4, 1, 1\rangle$ onto $\mathbf{n}$. This is given by

$$\frac{|\overrightarrow{QP} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{8}{\sqrt{6}} = \frac{4\sqrt{6}}{3}.$$

Problem 12. Determine whether the line with parametric equations

$$x = 1 + t, \quad y = 1 - t, \quad z = 2 + t, \quad t \in \mathbb{R}$$

intersects the plane with equation $x + 2y + z - 10 = 0$. If it does intersect, find the point of intersection.

Solution: Substituting the parametrization into the equation for the plane we obtain

$$0 = x + 2y + z - 10 = (1 + t) + 2(1 - t) + (2 + t) - 10 = -5.$$

Now $0 = -5$ has no solution. Thus the line does not intersect the plane.

Alternatively: The vector $\mathbf{v} = \langle 1, -1, 1\rangle$ is pointing in the direction of the line. The plane has normal vector $\mathbf{n} = \langle 1, 2, 1\rangle$. Since $\mathbf{v} \cdot \mathbf{n} = 0$, we conclude that the line is orthogonal to the normal vector. Thus the line is parallel to the plane. Either the line is contained in the plane or they do not intersect. The point $(1, 1, 2)$ is on the line, but is not on the plane so they do not intersect.

Problem 13. Evaluate $\lim_{t\to 1} \left\langle \frac{t^2 - 1}{t - 1}, \ln(t), e^{-t} \right\rangle$.

Solution: We find $\lim_{t\to 1} \frac{t^2 - 1}{t - 1} = \lim_{t\to 1} \frac{(t + 1)(t - 1)}{t - 1} = 2$ and know $\ln(1) = 0$. Thus, the limit is $\langle 2, 0, e^{-1}\rangle$.

Problem 14. Find the tangent vector to $\mathbf{r}(t) = \langle t^2, \ln(t), e^t\rangle$ at $t = 1$.

Solution: First, $\mathbf{r}'(t) = \langle 2t, \frac{1}{t}, e^t\rangle$. Then $\mathbf{r}'(1) = \langle 2, 1, e\rangle$.

Problem 15. Evaluate the integral $\int \left(t^2\mathbf{i} + \cos(t)\mathbf{j} - e^t\mathbf{k}\right) dt$.

Solution: We find $\frac{t^3}{3}\mathbf{i} + \sin(t)\mathbf{j} - e^t\mathbf{k} + \mathbf{C}$ for some constant vector $\mathbf{C}$.

Problem 16. Determine the length of the parametric curve given by $x = 3t^2$, $y = 2t^3$ where $0 \le t \le 1$.

Solution: The arc length formula gives us

$$\begin{aligned} s &= \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt \\ &= \int_0^1 \sqrt{(6t)^2 + (6t^2)^2}\ dt \\ &= \int_0^1 \sqrt{36t^2 + 36t^4}\ dt \\ &= \int_0^1 6t\sqrt{1+t^2}\ dt \end{aligned}$$

Use the substitution $u = 1 + t^2$, $du = 2t\ dt$:

$$= \int_1^2 3\sqrt{u}\ du = 2u^{\frac{3}{2}}\Big|_1^2 = 4\sqrt{2} - 2$$

Problem 17. Find the principal unit tangent vector and the principal unit normal vector for the curve

$$\mathbf{r}(t) = \langle \sin(t), \cos(t), 1 - 3t \rangle$$

at $t = 0$.

Solution: For the principal unit tangent vector, we first compute

$$\mathbf{r}'(t) = \langle \cos(t), -\sin(t), -3 \rangle$$

and $\|\mathbf{r}'(t)\| = \sqrt{\cos^2(t) + \sin^2(t) + 3^2} = \sqrt{10}$. Thus,

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \left\langle \frac{\cos(t)}{\sqrt{10}}, -\frac{\sin(t)}{\sqrt{10}}, -\frac{3}{\sqrt{10}} \right\rangle.$$

For the principal unit normal vector, we compute

$$\mathbf{T}'(t) = \left\langle -\frac{\sin(t)}{\sqrt{10}}, -\frac{\cos(t)}{\sqrt{10}}, 0 \right\rangle$$

and $\|\mathbf{T}'(t)\| = \frac{1}{\sqrt{10}}$. Thus

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \langle -\sin(t), -\cos(t), 0 \rangle.$$

Plugging in $t = 0$, we obtain

$$\mathbf{T}(0) = \left\langle \frac{1}{\sqrt{10}}, 0, -\frac{3}{\sqrt{10}} \right\rangle, \text{ and } \mathbf{N}(0) = \langle 0, -1, 0 \rangle.$$
