University of South Carolina

Midterm Examination 3 November 20, 2018

Math 142–H01

Closed book examination Time: 75 minutes

Name Solutions

Instructions:

No notes, books, or calculators are allowed. If you need more space than is provided use the back of the previous page and clearly indicate you have done so. Simplify your final answers. **Full credit may not be awarded for insufficient accompanying work.**

1	12	12
2	9	9
3	6	6
4	8	8
5	6	6
6	9	9
Total	50	50

1. (12 points) For each of the following functions:

- write down the Maclaurin series using Σ notation, and
- write down the radius of convergence.

(You do not need to justify your answers.)

(a) e^x

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad R=\infty$$

(b) $\cos(x)$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad R=\infty$$

(c) $(1+x)^{\frac{1}{3}}$

$$\sum_{n=0}^{\infty} \binom{1/3}{n} x^n \qquad R=1$$

(d) $\ln(1+x)$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} \qquad R=1$$

2. (9 points) For each of the following series, determine if it converges or diverges.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n^n}$ Root test: $\displaystyle\lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^n}} = \lim_{n\to\infty} \frac{2}{n} = 0$

$0<1$ so Converges

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{(2n)!}$ Ratio test: $\displaystyle\lim_{n\to\infty} \left| \frac{(n+1)+2}{(2(n+1))!} \Big/ \frac{n+2}{(2n)!} \right| = \lim_{n\to\infty} \frac{n+3}{n+2} \frac{(2n)!}{(2n+2)!}$

$\displaystyle = \lim_{n\to\infty} \frac{n+3}{(n+2)(2n+1)(2n+2)} = 0$

$0<1$ so Converges

(c) $\displaystyle\sum_{n=3}^{\infty} \frac{n^2+2n-1}{n^4-2n+3}$ Limit comparison with $\sum \frac{1}{n^2}$.

$\displaystyle\lim_{n\to\infty} \left| \left(\frac{n^2+2n-1}{n^4-2n+3}\right) \Big/ \left(\frac{1}{n^2}\right) \right| = \lim_{n\to\infty} \frac{n^4-2n^3-n^2}{n^4-2n+3} = 1$

Thus they have the same convergence.

Since $\sum \frac{1}{n^2}$ converges by p-series test, so does this.

Converges.

3. (6 points) Determine the Taylor polynomial of order 3 generated by the function $f(x) = \tan(x)$ at $a = \pi$.

$f^{(0)}(x) = \tan(x)$

$f'(x) = \sec^2(x)$

$f''(x) = 2\sec(x)\sec(x)\tan x = 2\sec^2(x)\tan(x)$

$f'''(x) = 4\sec(x)\sec(x)\tan(x)\tan(x) + 2\sec^2(x)\sec^2(x)$

$= 4\sec^2(x)\tan^2(x) + 2\sec^4(x)$

$f^{0}(\pi) = 0$

$f'(\pi) = 1$

$f^{(2)}(\pi) = 0$

$f^{(3)}(\pi) = 2$

Thus the Taylor polynomial is

$$P_3(x) = 1(x-\pi)^1 + \frac{2}{3!}(x-\pi)^3$$

$$= (x-\pi) + \frac{1}{3}(x-\pi)^3$$

4. (8 points) Determine the interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{(4x-5)^n}{3n+1}.$$

Ratio test: $\lim_{n\to\infty} \left| \frac{(4x-5)^{n+1}}{3(n+1)+1} \Big/ \frac{(4x-5)^n}{3n+1} \right|$

$= \lim_{n\to\infty} |4x-5| \left|\frac{3n+1}{3n+4}\right| = |4x-5|$

Converges when $|4x-5|<1$.

Solving: $-1 < 4x-5 < 1 \Rightarrow 4 < 4x < 6 \quad 1 < x < 3/2$

When $x=1$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+1}$ is alternating, $\sum \frac{1}{3n+1}$ is decreasing and $\lim_{n\to\infty} \frac{1}{3n+1} = 0 \Rightarrow$ Converges by Alternating series test

When $x=\frac{3}{2}$: $\sum_{n=1}^{\infty} \frac{1}{3n+1}$ Limit comparison with $\sum \frac{1}{n}$, which diverges.

$\lim_{n\to\infty} \frac{1}{3n+1} \Big/ \frac{1}{n} = \lim_{n\to\infty} \frac{n}{3n+1} = \frac{1}{3} \neq 0$ so both diverge.

The interval of convergence is $[1, 3/2)$.

5. (6 points)

(a) Estimate $\sqrt{6}$ using the Taylor polynomial of order 2 for $f(x) = \sqrt{x}$ at $a = 4$.

$f^{(0)}(x) = x^{1/2}$ $\qquad f^{(0)}(4) = 2$

$f^{(1)}(x) = \frac{1}{2}x^{-1/2}$ $\qquad f^{(1)}(4) = \frac{1}{2\times 2} = \frac{1}{4}$

$f^{(2)}(x) = -\frac{1}{4}x^{-3/2}$ $\qquad f^{(2)}(4) = -\frac{1}{4\times 2^3} = -\frac{1}{32}$

$$P_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{32}\frac{1}{2}(x-4)^2$$

$$P_2(6) = 2 + \frac{1}{4}2 - \frac{2^2}{32\times 2} = 2 + \frac{1}{2} - \frac{1}{16} = \frac{32+8-1}{16} = \frac{39}{16}$$

(b) Find an upper bound on the absolute value of the error for the estimate from (a) using the Remainder Estimation Theorem.

$|f^{(3)}(x)| = \left|\frac{3}{8}x^{-5/2}\right|$ is decreasing. On interval $[4,6]$ max is at $x=4$.

Thus $M = |f^{(3)}(4)| = \frac{3}{8}\cdot\frac{1}{2^5}$

$$R_2(6) \le \frac{M}{(2+1)!}|6-4|^{2+1} = \frac{1}{6}\frac{3}{8}\frac{1}{2^5}2^3 = \frac{1}{2}\frac{1}{8}\frac{1}{4} = \frac{1}{64}$$

6. (9 points) Find the following:

(a) The Taylor polynomial of order 34 generated by $f(x) = x^{10}\cos(x^5)$ at $a = 0$.

$$f(x) = x^{10}\left(1 - \frac{(x^5)^2}{2} + \frac{(x^5)^4}{4!} - \frac{(x^5)^6}{6!} + \cdots\right)$$

$$= x^{10} - \frac{x^{20}}{2} + \frac{x^{30}}{4!} - \frac{x^{40}}{6!} + \cdots$$

$$P_3(x) = x^{10} - \frac{x^{20}}{2} + \frac{x^{30}}{24}$$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^n}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n(\sqrt{2})^{2n}}{(2n)!} = \cos(\sqrt{2})$

(c) $\displaystyle\lim_{x\to 0} \frac{6\sin(x) - 6x + x^3}{x^5} = \lim_{x\to 0} \frac{6\left(x - x^3/6 + x^5/120 - \cdots\right) - 6x + x^3}{x^5}$

$$= \lim_{x\to 0} \frac{6x - x^3 + x^5/20 + \cdots - 6x - x^3}{x^5}$$

$$= \lim_{x\to 0} \frac{x^5/20 + \cdots}{x^5} = \frac{1}{20}$$

The End