University of South Carolina

Midterm Examination 3 November 21, 2017

Math 142–005/006

Closed book examination Time: 75 minutes

Name Solutions

Instructions:

No notes, books, or calculators are allowed. If you need more space than is provided use the back of the previous page and clearly indicate you have done so. Simplify your final answers. **Full credit may not be awarded for insufficient accompanying work.**

1	12	12
2	8	8
3	8	8
4	6	6
5	9	9
6	7	7
Total	50	50

1. (12 points) For each of the following functions:

- write down the Maclaurin series using Σ notation, and
- write down the radius of convergence.

(You do not need to justify your answers.)

(a) e^x

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad R = \infty$$

(b) $\cos(x)$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad R = \infty$$

(c) $(1+x)^{\frac{1}{2}}$

$$= \sum_{n=0}^{\infty} \binom{1/2}{n} x^n \qquad R = 1$$

(d) $\tan^{-1}(x)$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad R = 1$$

2. (8 points) Determine the Taylor polynomial of order 3 generated by the function $f(x) = \frac{1}{x^2}$ at $x = 1$.

$f^{(0)}(x) = x^{-2}$ $\quad f^{(0)}(1) = 1$

$f^{(1)}(x) = -2x^{-3}$ $\quad f^{(1)}(1) = -2$

$f^{(2)}(x) = 6x^{-4}$ $\quad f^{(2)}(1) = 6$

$f^{(3)}(x) = -24x^{-5}$ $\quad f^{(3)}(1) = -24$

$$P_3(x) = 1 - 2(x-1) + \frac{6}{2}(x-1)^2 - \frac{24}{6}(x-1)^3$$

$$= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3$$

3. (8 points) Determine the interval of convergence for the power series

$$(*)\sum_{n=1}^{\infty} \frac{(4x-5)^n}{n}.$$

Root test: $\lim_{n\to\infty} \sqrt[n]{\left|\frac{(4x-5)^n}{n}\right|} = |4x-5|$

Converges when $|4x-5|<1$, diverges when $|4x-5|>1$.

solve $|4x-5|<1$: $-1<4x-5<1 \Rightarrow 1<x<\frac{3}{2}$

When $x=1$: $(*)$ is $\sum_{n=1}^{\infty} \frac{(4-5)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

Converges since alternating harmonic series.

When $x=\frac{3}{2}$: $(*)$ is $\sum_{n=1}^{\infty} \frac{(4(\frac{3}{2})-5)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$

Diverges since harmonic series.

The interval of convergence is $\left[1, \frac{3}{2}\right)$

$\boxed{\text{Comment: I did not deduct marks for dropping absolute value brackets, but be careful!}}$

4. (6 points)

(a) Using the Maclaurin polynomial of order 4 for $f(x) = e^x$, estimate the value of $\frac{1}{e}$.

$$e^x \approx P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

$$e^{-1} \approx P_4(-1) = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}$$

$$= \frac{12 - 4 + 1}{24} = \frac{9}{24} = \frac{3}{8}$$

(b) What is the maximum value of $|f^{(5)}(x)|$ on the interval $[-1, 0]$?

$f^{(5)}(x) = e^x$

e^x is an increasing positive fcn so maximum is at $x = 0$.

$$M = |e^0| = 1$$

(c) Find an upper bound on the absolute value of the error for the estimate from (a) using the Remainder Estimation Theorem.

$$|R_4(-1)| \leq \frac{M\,(-1-0)^5}{5!} = \frac{1}{120}$$

5. (9 points) Find the following:

(a) The Taylor polynomial of order 3 generated by $f(x) = \ln(x+1)/x$ at $x = 0$.

$$\ln(x+1) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$P_3(x) = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4}$$

(b) The Taylor polynomial of order 9 generated by $f(x) = e^{-x^3}$ at $x = 0$.

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots$$

$$P_9(x) = 1 + (-x^3) + \frac{(-x^3)^2}{2} + \frac{(-x^3)^3}{6}$$

$$= 1 - x^3 + \frac{x^6}{2} - \frac{x^9}{6}$$

(c) $\lim_{x\to 0} \frac{\cos(x) - 1 + \frac{x^2}{2}}{x^4} = \lim_{x\to 0} \frac{\left(1 - x^2/2 + x^4/4! - \cdots\right) - \left(1 - \frac{x^2}{2}\right)}{x^4}$

$$= \lim_{x\to 0} \frac{\left(\frac{x^4}{4!} - \cdots\right)}{x^4} = \lim_{x\to 0}\left(\frac{1}{4!} - \cdots\right) = \frac{1}{24}$$

6. (7 points) Let C be the parametric curve determined by

$$x = t^2$$
$$y = t^3 + t$$

where t is a parameter in the interval $[0, 2]$.

(a) Determine the x and y coordinates of the point when $t = 1$.

$x = (1)^2 = 1$ $\qquad$ $y = (1)^3 + (1) = 2$

(b) Determine $\left.\frac{dy}{dx}\right|_{t=1}$.

$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2+1}{2t}$

$\frac{dx}{dt} = 2t$

$\left.\frac{dy}{dx}\right|_{t=1} = \frac{3(1)+1}{2} = 2$

$\frac{dy}{dt} = 3t^2 + 1$

(c) Find an equation for the line tangent to the curve C at the point where $t = 1$.

Slopepoint form: $(y - y_0) = m(x - x_0)$

$\Rightarrow \quad y - 2 = 2(x - 1)$

(d) Determine $\left.\frac{d^2y}{dx^2}\right|_{t=1}$.

$\left.\frac{d}{dt}(y')\right|_{t=1} = \left.\frac{(6t)(2t) - (3t^2+1)2}{(2t)^2}\right|_{t=1} = \frac{6\times 2 - 4\times 2}{4} = 1$

$\left.\frac{d^2y}{dx^2}\right|_{t=1} = \left.\frac{\frac{d}{dt}(y')}{dx/dt}\right|_{t=1} = \frac{1}{2}$

The End