University of South Carolina

Midterm Examination 3 November 21, 2017

Math 142–003/004

Closed book examination Time: 75 minutes

Name Solutions

Instructions:

No notes, books, or calculators are allowed. If you need more space than is provided use the back of the previous page and clearly indicate you have done so. Simplify your final answers. **Full credit may not be awarded for insufficient accompanying work.**

1	12	12
2	8	8
3	8	8
4	6	6
5	9	9
6	7	7
Total	50	50

1. (12 points) For each of the following functions:

- write down the Maclaurin series using Σ notation, and
- write down the radius of convergence.

(You do not need to justify your answers.)

(a) e^x

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad R = \infty$$

(b) $\cos(x)$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad R = \infty$$

(c) $(1+x)^{\frac{1}{2}}$

$$= \sum_{n=0}^{\infty} \binom{1/2}{n} x^n \qquad R = 1$$

(d) $\tan^{-1}(x)$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad R = 1$$

2. (8 points) Determine the Taylor polynomial of order 3 generated by the function $f(x) = \dfrac{1}{x}$ at $x = 2$.

$f^{(0)}(x) = x^{-1}$ $\qquad f^{(0)}(2) = \frac{1}{2}$

$f^{(1)}(x) = -x^{-2}$ $\qquad f^{(1)}(2) = -\frac{1}{4}$

$f^{(2)}(x) = 2x^{-3}$ $\qquad f^{(2)}(2) = \frac{1}{4}$

$f^{(3)}(x) = -6x^{-4}$ $\qquad f^{(3)}(2) = -\frac{3}{8}$

$$P_3(x) = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{4}\frac{1}{2}(x-2)^2 - \frac{3}{8}\frac{1}{6}(x-2)^3$$

$$= \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3$$

3. (8 points) Determine the interval of convergence for the power series

$$(*) \sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}.$$

Root test: $\lim_{n\to\infty} \sqrt[n]{\left|\frac{(3x-2)^n}{n}\right|} = |3x-2|$

Thus converges when $|3x-2|<1$, diverges when $|3x-2|>1$.

Solve $|3x-2|<1$: $-1<3x-2<1 \Rightarrow \frac{1}{3}<x<1$.

When $x=\frac{1}{3}$: $(*)$ is $\sum_{n=1}^{\infty} \frac{(3\frac{1}{3}-2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

(Converges since alternating harmonic series)

When $x=1$: $(*)$ is $\sum_{n=1}^{\infty} \frac{(3-2)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$

(Diverges since harmonic series)

Thus interval of convergence is $\left[\frac{1}{3}, 1\right)$.

Comment: I did not deduct marks for dropping absolute value brackets, but be careful!

4. (6 points)

(a) Using the Maclaurin polynomial of order 4 for $f(x) = e^x$, estimate the value of $\frac{1}{e}$.

$$e^x \approx P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

$$e^{-1} \approx P_4(-1) = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}$$

$$= \frac{12 - 4 + 1}{24} = \frac{9}{24} = \frac{3}{8}$$

(b) What is the maximum value of $|f^{(5)}(x)|$ on the interval $[-1, 0]$?

$f^{(5)}(x) = e^x$ e^x is an increasing positive function so maximum is at $x = 0$.

$$M = |e^0| = 1$$

(c) Find an upper bound on the absolute value of the error for the estimate from (a) using the Remainder Estimation Theorem.

$$|R_4(-1)| \leq \frac{M\,(-1-0)^5}{5!} = \frac{1}{120}$$

5. (9 points) Find the following:

(a) The Taylor polynomial of order 3 generated by $f(x) = \sin(2x)$ at $x = 0$.

$$\sin(z) = z - \frac{z^3}{3!} + \dots$$

Thus $P_3(x) = 2x - \frac{(2x)^3}{3!} = 2x - \frac{4}{3}x^3$

(b) The Taylor polynomial of order 5 generated by $f(x) = x^3e^x$ at $x = 0$.

$$e^x = 1 + x + \frac{x^2}{2!} + \dots$$

Thus $P_5(x) = x^3 + x^4 + \frac{x^5}{2}$

(c) $\lim\limits_{x\to 0} \frac{\sin(x) - x + \frac{x^3}{6}}{x^5} = \lim\limits_{x\to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right) - \left(x - \frac{x^3}{3!}\right)}{x^5}$

$$= \lim_{x\to 0} \frac{\frac{1}{5!}x^5 + \dots}{x^5} = \lim_{x\to 0}\left(\frac{1}{5!} + \dots\right) = \frac{1}{120}$$

6. (7 points) Let C be the parametric curve determined by

$$x = t^3$$
$$y = 3t^2 - t$$

where t is a parameter in the interval $[0, 2]$.

(a) Determine the x and y coordinates of the point when $t = 1$.

$x = (1)^3 = 1 \qquad y = 3(1) - 1 = 2$

(b) Determine $\left.\frac{dy}{dx}\right|_{t=1}$.

$\frac{dx}{dt} = 3t^2 \qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6t-1}{3t^2}$

$\frac{dy}{dt} = 6t - 1 \qquad y' = \frac{dy}{dx} = \frac{5}{3}$

(c) Find an equation for the line tangent to the curve C at the point where $t = 1$.

Slope point form: $(y - y_0) = m(x - x_0)$ becomes:

$(y-2) = \frac{5}{3}(x-1)$

(d) Determine $\left.\frac{d^2y}{dx^2}\right|_{t=1}$.

$\left.\frac{d}{dt}\left(\frac{dy}{dx}\right)\right|_{t=1} = \left.\frac{6(3t^2) - (6t-1)6t}{(3t^2)^2}\right|_{t=1} = \frac{6 \times 3 - 5 \times 6}{9} = -\frac{4}{3}$

$\left.\frac{d^2y}{dx^2}\right|_{t=1} = \left.\frac{\frac{d}{dt}(y')}{dx/dt}\right|_{t=1} = \frac{-4/3}{3} = -\frac{4}{9}$

The End