**University of South Carolina**

Midterm Examination 3 October 20, 2016

**Math 142 Section H03**

Closed book examination Time: 75 minutes

**Name** Solutions

**Instructions:**

No notes, books, or calculators are allowed. If you need more space than is provided use the back of the previous page and clearly indicate you have done so. Simplify your final answers. **Full credit may not be awarded for insufficient accompanying work.**

| | | |
|---|---|---|
| 1 | 8 | 8 |
| 2 | 8 | 8 |
| 3 | 8 | 8 |
| 4 | 10 | 10 |
| 5 | 8 | 8 |
| 6 | 8 | 8 |
| Total | 50 | 50 |

1. (8 points) For each of the following functions:

- write down the Maclaurin series using $\Sigma$ notation, and
- write down their interval of convergence.

(You do not need to justify your answers.)

(a) $e^x$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad x \in (-\infty, \infty)$$

(b) $\sin(x)$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad x \in (-\infty, \infty)$$

(c) $(1+x)^{\frac{1}{3}}$

$$\sum_{n=0}^{\infty} \binom{1/3}{n} x^n, \quad x \in (-1, 1)$$

(d) $\ln(1+x)$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad x \in (-1, 1]$$

2. (8 points) Determine the Taylor polynomial of order 4 generated by the function $\cos^2(x)$ at $x = \pi$.

$f(x) = \cos^2(x)$ $\quad f(\pi) = 1$

$f'(x) = -2\sin(x)\cos(x)$ $\quad f'(\pi) = 0$

$f''(x) = -2\cos^2(x) + 2\sin^2(x)$ $\quad f''(\pi) = -2$

$f'''(x) = 8\sin(x)\cos(x)$ $\quad f'''(\pi) = 0$

$f^{(4)}(x) = 8\cos^2(x) - 8\sin^2(x)$ $\quad f^{(4)}(\pi) = 8$

$$P_4(x) = 1 + 0(x-\pi) + \frac{(-2)}{2}(x-\pi)^2 + \frac{0}{3!}(x-\pi)^3 + \frac{8}{24}(x-\pi)^4$$

$$P_4(x) = 1 - (x-\pi)^2 + \frac{1}{3}(x-\pi)^4$$

3. (8 points) Determine the interval of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{(x-5)^n}{2n+1}.$$

Ratio test: $\lim_{n\to\infty} \left| \frac{(x-5)^{n+1}}{2(n+1)+1} \Big/ \frac{(x-5)^n}{2n+1} \right|$

$$= \lim_{n\to\infty} \frac{2n+1}{2n+3} |x-5| = |x-5|$$

Thus converges absolutely when $|x-5|<1$ or $4<x<6$

diverges when $|x-5|>1$. Inconclusive when $x=4$ or $6$.

For $x=4$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges by alternating series test

For $x=6$: $\sum_{n=0}^{\infty} \frac{1^n}{2n+1}$ diverges by limit comparison test with $\frac{1}{n}$:

$$\lim_{n\to\infty} \frac{1}{2n+1} \Big/ \frac{1}{n} = \frac{1}{2}.$$

Thus interval of convergence is $x \in [4,6)$

4. (10 points)

(a) Using the Taylor polynomial of order 2 generated by the function $f(x) = \sqrt{x}$ at $x = 4$, estimate the value of $\sqrt{5}$.

$f(x) = \sqrt{x}$ $\quad f(4) = 2$

$f'(x) = \frac{1}{2}x^{-1/2}$ $\quad f'(4) = \frac{1}{4}$

$f''(x) = -\frac{1}{4}x^{-3/2}$ $\quad f''(4) = -\frac{1}{32}$

$$P_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$$

$$P_2(5) = 2 + \frac{1}{4} - \frac{1}{64} = \frac{143}{64}$$

(b) What is the maximum value of $|f^{(3)}(x)|$ on the interval $[4, 5]$?

$f'''(x) = \frac{3}{8}x^{-5/2}$ is decreasing. maximum is $f'''(4) = \frac{3}{8 * 2^5} = \frac{3}{256}$

(c) Find an upper bound on the absolute value of the error for the estimate from (a) using the Remainder Estimation Theorem.

Use $M = \frac{3}{256}$ in $|R_2(x)| \leq \frac{M|x-a|^3}{3!}$

$$|R_2(5)| \leq \frac{(3/256) \times 1}{6} = \frac{1}{512}$$

5. (8 points) Find the following:

(a) $\lim_{x\to 0} \frac{\tan^{-1}(x) - x}{\sin(x) - x}$

$$= \lim_{x\to 0} \frac{\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right) - x}{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) - x}$$

$$= \lim_{x\to 0} \frac{-\frac{x^3}{3} + \frac{x^5}{5} - \cdots}{-\frac{x^3}{3!} + \frac{x^5}{5!} - \cdots} = \lim_{x\to 0} \frac{-1/3 + x^2/5 \cdots}{-1/6 + x^2/5 \cdots}$$

$$= 2$$

(b) $\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$

(c) $\sum_{n=0}^{\infty} \frac{4^n}{n!} = e^4$ since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

6. (8 points) Use the Taylor polynomial of order 3 generated by $\sin(x)$ at $x = 0$ to estimate

$$\int_0^3 \frac{\sin(2x)}{x}\, dx\ .$$

Taylor polynomial of $\sin(x)$ is $x - \frac{x^3}{3!}$

$$\int_0^3 \frac{\sin(2x)}{x}\,dx \approx \int_0^3 \frac{(2x) - \frac{(2x)^3}{3!}}{x}\,dx$$

$$= \int_0^3 2 - \frac{2^3}{3!}x^2\,dx$$

$$= \left[2x - \frac{1}{3}\frac{2^3}{3!}x^3\right]_0^3 = 2(3) - \frac{1}{3}\frac{2^3}{3\times 2}3^3$$

$$= -6$$

**The End**