Problem 1 For each series, what can you conclude from the given convergence test?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n^2}$ using the Root Test.

Solution: We evaluate

$$\rho = \lim_{n\to\infty} \left|\frac{5^n}{n^2}\right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{5}{n^{2/n}} = \frac{5}{\left(\lim_{n\to\infty} n^{1/n}\right)^2} = \frac{5}{1^2} = 5 \ .$$

Since $\rho > 1$, the series diverges by the Root Test.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ using the Ratio Test.

Solution: We evaluate

$$\rho = \lim_{n\to\infty} \left|\frac{2^{(n+1)}}{(n+1)!}\Big/\frac{2^n}{n!}\right| = \lim_{n\to\infty} \frac{3^{n+1}n!}{2^n(n+1)!} = \lim_{n\to\infty} \frac{2n!}{(n+1)\times n!} = \lim_{n\to\infty} \frac{2}{n+1} = 0 \ .$$

Since $\rho < 1$, the series converges by the Ratio Test.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2}{1+n^3}$ using the Integral Test.

Solution: Using a substitution $u = 1 + x^3$ with $du = 3x^2\ dx$, we find that

$$\int_1^{\infty} \frac{3x^2}{1+x^3}\ dx = \lim_{b\to\infty} \int_1^b \frac{3x^2}{1+x^3}\ dx = \lim_{b\to\infty} \int_2^{1+b^3} \frac{1}{u}\ du = \lim_{b\to\infty} \left[\ln|u|\right]_2^{1+b^3} = \lim_{b\to\infty} \big(\ln(1+b^3) - \ln(2)\big) = \infty \ .$$

Since the improper integral diverges, so must the series by Integral Test.

Comments: Be careful with your bounds! You must change the bounds of your integral along with a u-substitution. It turns out not to matter here, but I deducted marks if you didn't carry out the process correctly.

Problem 2 For each series, what can you conclude from the given convergence test?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2+n}$ using the Limit Comparison Test with $\displaystyle\sum \frac{1}{n^2}$.

Solution: We find that

$$L = \lim_{n\to\infty} \left(\frac{\ln(n)}{n^2+n} \Big/ \frac{1}{n^2} \right) = \lim_{n\to\infty} \frac{\ln(n)n^2}{n^2+n} = \lim_{n\to\infty} \frac{\ln(n)}{1+\frac{1}{n}} = \infty \ .$$

Since $\displaystyle\sum \frac{1}{n^2}$ converges (p-series with $p=2$), the Limit Comparison Test is inconclusive.

(b) $\displaystyle\sum_{n=4}^{\infty} \frac{n+3n^2}{n^3+4}$ using the Limit Comparison Test with $\displaystyle\sum \frac{1}{n}$.

Solution: We find that

$$L = \lim_{n\to\infty} \left(\frac{n+3n^2}{n^3+4} \Big/ \frac{1}{n} \right) = \lim_{n\to\infty} \frac{n^2+3n^3}{n^3+4} = 3 \ .$$

Since $\displaystyle\sum \frac{1}{n}$ diverges (p-series with $p=2$), the Limit Comparison Test tells us the series diverges.

(c) $\displaystyle\sum_{n=4}^{\infty} \frac{n+2}{2^n}$ using the Limit Comparison Test with $\displaystyle\sum \frac{1}{n^2}$.

Solution: We find that

$$L = \lim_{n\to\infty} \left(\frac{n+2}{2^n} \Big/ \frac{1}{n^2} \right) = \lim_{n\to\infty} \frac{n^3+2n^2}{2^n} = \lim_{n\to\infty} \frac{3n^2+4n}{\ln(2)2^n} = \lim_{n\to\infty} \frac{6n+4}{\ln(2)^2 2^n} = \lim_{n\to\infty} \frac{6}{\ln(2)^3 2^n} = 0 \ .$$

Since $\displaystyle\sum \frac{1}{n^2}$ converges (p-series with $p=2$), the Limit Comparison Test tells us the series converges.

Problem 3 For each of the following series, determine if it converges or diverges.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{3^{3n}}{(1+n)^n}$

Solution: We apply the root test

$$\rho = \lim_{n\to\infty} \left| \frac{3^{3n}}{(1+n)^n} \right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{3^3}{1+n} = 0 \ .$$

Since $\rho < 1$, the series converges.

Comments: Ratio test will work, but is much more complicated:

$$\rho = \lim_{n\to\infty} \left| \frac{3^{3(n+1)}}{((n+1)+1)^{n+1}} \Big/ \frac{3^{3n}}{(n+1)^n} \right| = \lim_{n\to\infty} \frac{3^{3n+3}(n+1)^n}{3^{3n}(n+2)^{n+1}} = \lim_{n\to\infty} \frac{27}{n+2} \left(\frac{n+1}{n+2} \right)^n$$

Now you have a very hard problem with l'Hôpital's rule. Or you can remember that $\displaystyle\lim_{n\to\infty} \left(1+\frac{r}{n}\right)^n = e^r$, and continue:

$$\rho = \lim_{n\to\infty} \frac{27(1+\frac{1}{n})^n}{(n+2)(1+\frac{2}{n})^n} = \lim_{n\to\infty} \frac{27e}{(n+2)e^2} = 0$$

Integral test will not work since the antiderivative cannot be found using functions we've seen in the course.

(b) $\displaystyle\sum_{n=3}^{\infty} \frac{n^2+2}{2n^4-1}$

Solution: We apply the limit comparison test with $\displaystyle\sum \frac{1}{n^2}$ (which converges: p-series with $p = 2$).

$$L = \lim_{n\to\infty} \left(\frac{n^2+2}{2n^4-1} \Big/ \frac{1}{n^2} \right) = \lim_{n\to\infty} \frac{n^4+2n^2}{2n^4-1} = \frac{1}{2} \ .$$

By the limit comparison test, we conclude the series converges.

Comments: The root and ratio test will be inconclusive. The integral test becomes a very tedious partial fractions problem.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{(2n+1)!}$

Solution: We evaluate

$$\rho = \lim_{n\to\infty} \left| \frac{4^{n+1}}{(2(n+1)+1)!} \middle/ \frac{4^n}{(2n+1)!} \right| = \lim_{n\to\infty} \frac{4^{n+1}(2n+1)!}{4^n(2n+3)!} = \lim_{n\to\infty} \frac{4}{(2n+3)(2n+2)} = 0\ .$$

Since $\rho < 1$, the series converges by the Ratio Test.

Comments: Most of the other tests are useless since factorials are present.

Problem 4 For each of the following series, determine if it

- converges absolutely,
- converges conditionally, or
- diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$

Solution: Taking the absolute values of the terms produces the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$. We use the limit comparison test with the convergent series $\sum \frac{1}{n^2}$:

$$L = \lim_{n\to\infty} \left(\frac{1}{n^2+1} \middle/ \frac{1}{n^2} \right) = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 \ .$$

Thus the series with positive terms is convergent. Thus the original given series is absolutely convergent.

Comments: The alternating series test will tell us that the series is convergent, but will not tell us anything about absolute convergence. Since the series turns out to be absolutely convergent, we don't need the alternating series test. Note that both comparison tests and the integral test do not apply to the original series with the $(-1)^n$ present. You can only apply them to the corresponding series with positive terms. Ratio test and root test will be inconclusive here.

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}n}{n-1}$

Solution: Observe that $\lim_{n\to\infty} \frac{n}{n-1} = 1$ so the limit $\lim_{n\to\infty}(-1)^{n-1}\frac{n}{n-1}$ cannot exist. Thus the series diverges by the divergence test.

Comments: Technically, the alternating series test cannot be used to show divergence, but I let it slide since you're essentially finding the same limit anyway. Note that the corresponding sequence $\{\frac{n}{n-1}\}$ is non-increasing so this piece of the alternating series test is okay. All the remaining tests only work for series with positive terms, so they can only exclude absolute convergence — the divergence test is the only way to do this problem.

Problem 5 Determine the interval of convergence for the power series

$$\sum_{n=2}^{\infty} \frac{n(2x-1)^n}{4^n}.$$

Solution: First, we apply the root test:

$$\rho = \lim_{n\to\infty} \left| \frac{n(2x-1)^n}{4^n} \right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^{1/n}|2x-1|}{4} = \frac{|2x-1|}{4} \ .$$

Thus, the series converges when $\frac{|2x-1|}{4} < 1$ and diverges when $\frac{|2x-1|}{4} > 1$. The expression $\frac{|2x-1|}{4} < 1$ can be rewritten as

$$\begin{aligned} -1 <& \frac{2x-1}{4} < 1 \\ -4 <& \ 2x-1 < 4 \\ -3 <& \quad 2x < 5 \\ -\frac{3}{2} <& \quad\ x < \frac{5}{2} \ . \end{aligned}$$

It remains to determine convergence at the special points $x = -\frac{3}{2}$ and $\frac{5}{2}$. Wwen $-\frac{3}{2}$, the series becomes

$$\sum_{n=2}^{\infty} \frac{n(2(-\frac{3}{2})-1)^n}{4^n} = \sum_{n=2}^{\infty} \frac{n(-4)^n}{4^n} = \sum_{n=2}^{\infty} n(-1)^n,$$

which diverges by divergence test. When $x = \frac{5}{2}$, the series becomes

$$\sum_{n=2}^{\infty} \frac{n(2(\frac{5}{2})-1)^n}{4^n} = \sum_{n=2}^{\infty} \frac{n4^n}{4^n} = \sum_{n=2}^{\infty} n,$$

which diverges by divergence test. Thus the interval of convergence is $(-\frac{3}{2}, \frac{5}{2})$.

Problem 6 The power series

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$
$$g(x) = \sum_{n=0}^{\infty} nx^n = x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \cdots$$

converge for $-1 < x < 1$ (you don't need to show this). Find the first 4 non-zero terms of the following power series:

(a) $f(x) - g(x)$

Solution: We find

$$\begin{aligned} f(x) + g(x) &= (1 + x + x^2 + x^3 + x^4 + \cdots) - (x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \cdots) \\ &= (1 - 0) + (x - x) + (x^2 - 2x^2) + (x^3 - 3x^3) + (x^4 - 4x^4) + \cdots \\ &= 1 + 0x - x^2 - 2x^3 - 3x^4 \cdots \end{aligned}$$

(b) $f\left(\frac{x^3}{2}\right)$

Solution: We find

$$\begin{aligned} f\left(\frac{x^3}{2}\right) &= 1 + \left(\frac{x^3}{2}\right) + \left(\frac{x^3}{2}\right)^2 + \left(\frac{x^3}{2}\right)^3 + \left(\frac{x^3}{2}\right)^4 + \cdots \\ &= 1 + \frac{x^3}{2} + \frac{x^6}{4} + \frac{x^9}{8} + \cdots \end{aligned}$$

(c) $f'(x)$

Solution: We find

$$\begin{aligned} f'(x) &= \frac{d}{dx}(1 + x + x^2 + x^3 + x^4 + \cdots) \\ &= \frac{d}{dx}(1) + \frac{d}{dx}(x) + \frac{d}{dx}(x^2) + \frac{d}{dx}(x^3) + \frac{d}{dx}(x^4) + \cdots \\ &= 0 + 1 + 2x + 3x^2 + 4x^3 + \cdots \end{aligned}$$

(d) $\displaystyle\int_0^x f(y)\ dy$

Solution: We find

$$\begin{aligned}\int_0^x f(y)\ dy &= \int_0^x (1 + y + y^2 + y^3 + y^4 + \cdots)\ dy \\ &= \int_0^x 1\ dy + \int_0^x y\ dy + \int_0^x y^2\ dy + \int_0^x y^3\ dy + \int_0^x y^4\ dy + \cdots \\ &= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\end{aligned}$$