Problem 1 For each series, what can you conclude from the given convergence test?

(a) $\sum_{n=1}^{\infty} \frac{3^n}{n^2}$ using the Root Test.

Solution: We evaluate

$$\rho = \lim_{n\to\infty} \left|\frac{3^n}{n^2}\right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{3}{n^{2/n}} = \frac{3}{\left(\lim_{n\to\infty} n^{1/n}\right)^2} = \frac{3}{1^2} = 3 \ .$$

Since $\rho > 1$, the series diverges by the Root Test.

(b) $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ using the Ratio Test.

Solution: We evaluate

$$\rho = \lim_{n\to\infty} \left|\frac{3^{(n+1)}}{(n+1)!}\middle/\frac{3^n}{n!}\right| = \lim_{n\to\infty} \frac{3^{n+1}n!}{3^n(n+1)!} = \lim_{n\to\infty} \frac{3n!}{(n+1)\times n!} = \lim_{n\to\infty} \frac{3}{n+1} = 0 \ .$$

Since $\rho < 1$, the series converges by the Ratio Test.

(c) $\sum_{n=1}^{\infty} \frac{2n}{1+n^2}$ using the Integral Test.

Solution: Using a substitution $u = 1 + x^2$ with $du = 2x\ dx$, we find that

$$\int_1^\infty \frac{2x}{1+x^2}\ dx = \lim_{b\to\infty} \int_1^b \frac{2x}{1+x^2}\ dx = \lim_{b\to\infty} \int_2^{1+b^2} \frac{1}{u}\ du = \lim_{b\to\infty} \left[\ln|u|\right]_2^{1+b^2} = \lim_{b\to\infty} \left(\ln(1+b^2) - \ln(2)\right) = \infty \ .$$

Since the improper integral diverges, so must the series by Integral Test.

Comments: Be careful with your bounds! You must change the bounds of your integral along with a u-substitution. It turns out not to matter here, but I deducted marks if you didn't carry out the process correctly.

Problem 2 For each series, what can you conclude from the given convergence test?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2+n}$ using the Limit Comparison Test with $\displaystyle\sum \frac{1}{n}$.

Solution: We find that

$$L = \lim_{n\to\infty} \left(\frac{\ln(n)}{n^2+n} \Big/ \frac{1}{n} \right) = \lim_{n\to\infty} \frac{\ln(n)n}{n^2+n} = \lim_{n\to\infty} \frac{\ln(n)}{n+1} .$$

This is an indeterminate form. Applying l'Hôpital's Rule, we continue

$$L = \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{1} = 0.$$

Since $\displaystyle\sum \frac{1}{n}$ diverges (harmonic series), the Limit Comparison Test is inconclusive.

(b) $\displaystyle\sum_{n=4}^{\infty} \frac{2n+1}{n^3+1}$ using the Limit Comparison Test with $\displaystyle\sum \frac{1}{n^2}$.

Solution: We find that

$$L = \lim_{n\to\infty} \left(\frac{2n+1}{n^3+1} \Big/ \frac{1}{n^2} \right) = \lim_{n\to\infty} \frac{2n^3+n^2}{n^3+1} = 2 .$$

Since $\displaystyle\sum \frac{1}{n^2}$ converges (p-series with $p=2$), the Limit Comparison Test tells us the series converges.

(c) $\displaystyle\sum_{n=4}^{\infty} \frac{3n^2+1}{n^2+2}$ using the Limit Comparison Test with $\displaystyle\sum \frac{1}{n^2}$.

Solution: We find that

$$L = \lim_{n\to\infty} \left(\frac{3n^2+1}{n^2+2} \Big/ \frac{1}{n^2} \right) = \lim_{n\to\infty} \frac{3n^4+n^2}{n^2+2} = \infty .$$

Since $\displaystyle\sum \frac{1}{n^2}$ converges (p-series with $p=2$), the Limit Comparison Test is inconclusive.

Problem 3 For each of the following series, determine if it converges or diverges.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{2n}}{(n+1)^n}$

Solution: We apply the root test

$$\rho = \lim_{n\to\infty} \left|\frac{2^{2n}}{(n+1)^n}\right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{2^2}{n+1} = 0 \ .$$

Since $\rho < 1$, the series converges.

Comments: Ratio test will work, but is much more complicated:

$$\rho = \lim_{n\to\infty} \left|\frac{2^{2(n+1)}}{((n+1)+1)^{n+1}} \Big/ \frac{2^{2n}}{(n+1)^n}\right| = \lim_{n\to\infty} \frac{2^{2n+2}(n+1)^n}{2^{2n}(n+2)^{n+1}} = \lim_{n\to\infty} \frac{4}{n+2}\left(\frac{n+1}{n+2}\right)^n$$

Now you have a very hard problem with l'Hôpital's rule. Or you can remember that $\displaystyle\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r$, and continue:

$$\rho = \lim_{n\to\infty} \frac{4(1+\frac{1}{n})^n}{(n+2)(1+\frac{2}{n})^n} = \lim_{n\to\infty} \frac{4e}{(n+2)e^2} = 0$$

Integral test will not work since the antiderivative cannot be found using functions we've seen in the course.

(b) $\displaystyle\sum_{n=3}^{\infty} \frac{3n^2+1}{n^4-4}$

Solution: We apply the limit comparison test with $\displaystyle\sum \frac{1}{n^2}$ (which converges: p-series with $p = 2$).

$$L = \lim_{n\to\infty} \left(\frac{3n^2+1}{n^4-4} \Big/ \frac{1}{n^2}\right) = \lim_{n\to\infty} \frac{3n^4+n^2}{n^4-4} = 3 \ .$$

By the limit comparison test, we conclude the series converges.

Comments: The root and ratio test will be inconclusive. The integral test becomes a very tedious partial fractions problem.

(c) $\sum_{n=1}^{\infty} \frac{5^n}{(2n)!}$

Solution: We evaluate

$$\rho = \lim_{n\to\infty} \left| \frac{5^{n+1}}{(2(n+1))!} \Big/ \frac{5^n}{(2n)!} \right| = \lim_{n\to\infty} \frac{5^{n+1}(2n)!}{5^n(2n+2)!} = \lim_{n\to\infty} \frac{5}{(2n+2)(2n+1)} = 0\ .$$

Since $\rho < 1$, the series converges by the Ratio Test.

Comments: Most of the other tests are useless since factorials are present.

Problem 4 For each of the following series, determine if it

- converges absolutely,
- converges conditionally, or
- diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1}$

Solution: The series has terms $a_n = (-1)^n b_n$ where $b_n = \frac{1}{n+1}$ are all positive. Moreover, b_n is a decreasing sequence since $f(x) = \frac{1}{x+1}$ is a decreasing function for positive x. Finally $\lim_{n\to\infty} b_n = 0$. Thus the series is convergent by the alternating series test.

Taking the absolute values of the terms produces the series $\sum_{n=1}^{\infty} \frac{1}{n+1}$. We use the limit comparison test with the divergent series $\sum \frac{1}{n}$:

$$L = \lim_{n\to\infty} \left(\frac{1}{n+1} \bigg/ \frac{1}{n} \right) = \lim_{n\to\infty} \frac{n}{n+1} = 1\ .$$

Thus the series with positive terms is divergent.

Thus the original series is conditionally convergent.

Comments: The alternating series test is the only one that we know that will show the sequence is convergent. To show the series $\sum_{n=1}^{\infty} \frac{1}{n+1}$ divergent, we cannot use direct comparison with $\sum \frac{1}{n}$ since the inequality is the wrong direction. The root and ratio test are useless on $\sum_{n=1}^{\infty} \frac{1}{n+1}$.

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1} n}{n+1}$

Solution: Observe that $\lim_{n\to\infty} \frac{n}{n+1} = 1$ so the limit $\lim_{n\to\infty} (-1)^{n+1} \frac{n}{n+1}$ cannot exist. Thus the series diverges by the divergence test.

Comments: Technically, the alternating series test cannot be used to show divergence, but I let it slide since you're essentially finding the same limit anyway. The corresponding sequence $\{\frac{n}{n+1}\}$ is increasing so there's another reason the the alternating series test does not apply. All the remaining tests only work for series with positive terms, so they can only exclude absolute convergence — the divergence test is the only way to do this problem.

Problem 5 Determine the interval of convergence for the power series

$$\sum_{n=2}^{\infty} \frac{(x-3)^n}{n5^n}.$$

Solution: First, we apply the root test:

$$\rho = \lim_{n\to\infty} \left| \frac{(x-3)^n}{n5^n} \right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{|x-3|}{n^{1/n}5} = \frac{|x-3|}{5} \ .$$

Thus, the series converges when $\frac{|x-3|}{5} < 1$ and diverges when $\frac{|x-3|}{5} > 1$. The expression $\frac{|x-3|}{5} < 1$ can be rewritten as

$$\begin{aligned} -1 <& \frac{x-3}{5} < 1 \\ -5 <& \ x-3 < 5 \\ -2 <& \qquad x < 8 \ . \end{aligned}$$

It remains to determine convergence at the special points $x = -2$ and $x = 8$. Wwen $x = -2$, the series becomes

$$\sum_{n=2}^{\infty} \frac{((-2)-3)^n}{n5^n} = \sum_{n=2}^{\infty} \frac{(-5)^n}{n5^n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n},$$

which converges since it is the alternating harmonic series. When $x = 8$, the series becomes

$$\sum_{n=2}^{\infty} \frac{((8)-3)^n}{n5^n} = \sum_{n=2}^{\infty} \frac{(5)^n}{n5^n} = \sum_{n=2}^{\infty} \frac{1}{n},$$

which diverges since it is the harmonic series. Thus the interval of convergence is $[-2, 8)$.

Problem 6 The power series

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$
$$g(x) = \sum_{n=0}^{\infty} nx^n = x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \cdots$$

converge for $-1 < x < 1$ (you don't need to show this). Find the first 4 non-zero terms of the following power series:

(a) $f(x) + g(x)$

Solution: We find

$$\begin{aligned} f(x) + g(x) &= (1 + x + x^2 + x^3 + x^4 + \cdots) + (x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \cdots) \\ &= (1 + 0) + (x + x) + (x^2 + 2x^2) + (x^3 + 3x^3) + \cdots \\ &= 1 + 2x + 3x^2 + 4x^3 + \cdots \end{aligned}$$

(b) $g(2x^2)$

Solution: We find

$$\begin{aligned} g(2x^2) &= (2x^2) + 2(2x^2)^2 + 3(2x^2)^3 + 4(2x^2)^4 + \cdots \\ &= 2x^2 + 8x^4 + 24x^6 + 64x^8 + \cdots \end{aligned}$$

(c) $f'(x)$

Solution: We find

$$\begin{aligned} f'(x) &= \frac{d}{dx}(1 + x + x^2 + x^3 + x^4 + \cdots) \\ &= \frac{d}{dx}(1) + \frac{d}{dx}(x) + \frac{d}{dx}(x^2) + \frac{d}{dx}(x^3) + \frac{d}{dx}(x^4) + \cdots \\ &= 0 + 1 + 2x + 3x^2 + 4x^3 + \cdots \end{aligned}$$

(d) $\int_0^x f(y)\, dy$

Solution: We find

$$\begin{aligned}\int_0^x f(y)\ dy &= \int_0^x (1 + y + y^2 + y^3 + y^4 + \cdots)\ dy \\ &= \int_0^x 1\ dy + \int_0^x y\ dy + \int_0^x y^2\ dy + \int_0^x y^3\ dy + \int_0^x y^4\ dy + \cdots \\ &= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\end{aligned}$$