University of South Carolina

Midterm Examination 2 November 5, 2020

Math 142–011/012/031/032

Closed book examination Time: 75 minutes

Name Solutions

Instructions:

No notes, books, computer, phones, calculators or other aids are allowed. You must be alone while you are taking the test and you should not be in contact electronically or physically with any other person from the time you start the exam until you submit your final solutions. Do not use more time than the time allotted. Full credit will not be awarded for insufficient accompanying work.

Submit your completed exam on Blackboard by uploading a scanned PDF file or as multiple JPEG image files. You do not have to print the exam, but please have a separate page/image for each page of the exam. Clearly indicate which questions you are answering on every image/page.

1	12	12
2	12	12
3	12	12
4	8	8
5	8	8
6	8	8
Total	60	60

1. (12 points) For each series, what can you conclude from the given convergence test?

(a) $\displaystyle\sum_{n=4}^{\infty} \frac{1}{\ln(n)}$ using the Limit Comparison Test with $\sum \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{\frac{1}{\ln(n)}}{1/n} = \lim_{n\to\infty} \frac{n}{\ln(n)} = \lim_{n\to\infty} \frac{1}{(1/n)} = \lim_{n\to\infty} n = \infty$$

since $\sum \frac{1}{n}$ diverges, so does $\sum \frac{1}{\ln(n)}$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ using the Integral Test.

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx \quad \boxed{\begin{array}{l} u = \ln(x) \\ du = 1/x\,dx \end{array}} = \int_{\ln(2)}^{\infty} \frac{du}{u^2} = \lim_{b\to\infty} [-b^{-1}] - (-\ln(2))^{-1}$$

$$= 1/\ln(2) < \infty$$

Converges

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4n+3}{3n-5}\right)^n$ using the Root Test.

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{4n+3}{3n-5}\right|^n} = \lim_{n\to\infty} \frac{4n+3}{3n-5} = \frac{4}{3} > 1$$

Diverges

2. (12 points) For each series, what can you conclude from the given convergence test?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n+1}}{n3^{n-1}}$ using the Ratio Test.

$$\lim_{n\to\infty} \frac{2^{n+2}}{(n+1)3^{n}} \times \frac{n\,3^{n-1}}{2^{n+1}} = \lim_{n\to\infty} \frac{2}{3}\left(\frac{n}{n+1}\right) = \frac{2}{3} < 1$$

Converges

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{10^n}{n+1}$ using the Alternating Series Test.

$$\lim_{n\to\infty} \frac{10^n}{n+1} = \lim_{n\to\infty} \frac{\ln(10)\,10^n}{1} = \infty.$$

Thus Alternating series test does not apply.

[I accepted "diverges" here, which is true but does not follow from the AST.]

(c) $\displaystyle\sum_{n=10}^{\infty} \frac{n+1}{2n^3-2}$ using the Direct comparision Test with $\sum \frac{1}{n^2}$.

$$\frac{n+1}{2n^3-2} \overset{?}{<} \frac{1}{n^2}$$

$$n^2(n+1) \overset{?}{<} 2n^3-2$$

$$n^2+2 \overset{?}{<} n^3$$

Yes

Since $\sum \frac{1}{n^2}$ converges so does $\sum \frac{n+1}{2n^3-2}$

3. (12 points) For each of the following series, determine if it converges or diverges.

(a) $\displaystyle\sum_{n=3}^{\infty} \frac{n}{\ln(n)}$ nth term divergence

$$\lim_{n\to\infty} \frac{n}{(\ln n)} = \lim_{n\to\infty} \frac{1}{1/n} = \lim_{n\to\infty} n = \infty \neq 0$$

Diverges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n+1}}{(2n+1)!}$ Ratio test.

$$\rho = \lim_{n\to\infty} \frac{3^{n+2}}{(2(n+1)+1)!} \times \frac{(2n+1)!}{3^{n+1}} = \lim_{n\to\infty} \frac{3}{(2n+3)(2n+2)} = 0$$

Converges.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3+4^n}$ $\left(\frac{1}{2}\right)^n$ converges (geometric series)

Direct comparison test

$$\frac{2^n}{3+4^n} < \frac{2^n}{4^n} = \left(\frac{1}{2}\right)^n.$$

Converges

Alternate solutions to 3(c)

Limit comparison with $\sum \left(\frac{1}{2}\right)^n$ which converges.

$$\lim_{n\to\infty} \frac{\left(\frac{2^n}{3+4^n}\right)}{\left(\frac{1}{2}\right)^n} = \lim_{n\to\infty} \frac{2^n \times 2^n}{3+4^n} = \lim_{n\to\infty} \frac{1}{3/4^n+1} = 1$$

Converges.

Ratio test

$$\rho = \lim_{n\to\infty} \left(\frac{2^{n+1}}{3+4^{n+1}}\right) \times \left(\frac{3+4^n}{2^n}\right) = \lim_{n\to\infty} 2\left(\frac{3/4^n+1}{3/4^n+4}\right) = \frac{1}{2}$$

$\Rightarrow$ Converges

Root test

$$\rho = \lim_{n\to\infty} \sqrt[n]{\frac{2^n}{3+4^n}} = \lim_{n\to\infty} \frac{2}{(3+4^n)^{1/n}}$$

This is complex! That's why I did not give much partial credit for root test

Now $L = \lim_{n\to\infty} (3+4^n)^{1/n}$ gives

$$\ln L = \lim_{n\to\infty} \frac{\ln(3+4^n)}{n} = \lim_{n\to\infty} \frac{\ln(4)4^n}{3+4^n} = \ln(4) \lim_{n\to\infty} \frac{1}{3/4^n+1}$$

$$= \ln(4)$$

Thus $L=4$ and $\rho = \frac{2}{4} = \frac{1}{2}$.

$\Rightarrow$ Converges

4. (8 points) Determine the interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^3 3^n}.$$

Root test

$$\rho = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{x-1}{n^3 3^n}\right)^n\right|} = \lim_{n\to\infty} \frac{|x-1|}{(n^{1/n})^3\, 3} = \frac{|x-1|}{3}$$

Converges when $\frac{|x-1|}{3} < 1$. Solve $-1 < \frac{x-1}{3} < 1$.

$$-3 < x-1 < 3$$

Diverges when $\frac{|x-1|}{3} > 1$.

$$\boxed{-2 < x < 4.}$$

When $x=4$: $\sum_{n=1}^{\infty} \frac{(4-1)^n}{n^3 3^n} = \sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (p-series)

When $x=-2$: $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^3 (3)^n} = \sum_{n=1} \frac{(-1)^n}{n^3}$ converges (since $\sum \frac{1}{n^3}$ converges)

Interval of convergence is $-2 \leq x \leq 4$.

5. (8 points) Determine the Taylor polynomial of order 4 generated by $f(x) = \cos^2(x)$ at $x = 0$.

$f(x) = \cos^2(x)$

$f'(x) = -2\cos(x)\sin(x)$

$f''(x) = 2\sin^2(x) - 2\cos^2(x)$

$f'''(x) = 4\cos(x)\sin(x) + 4\cos(x)\sin(x)$

$= 8\cos(x)\sin(x)$

$f^{(4)}(x) = -8\sin^2(x) + 8\cos^2(x)$

$f(0) = 1$

$f'(0) = 0$

$f''(0) = -2$

$f'''(0) = 0$

$f^{(4)}(0) = 8$

$$P_4 = 1 + 0x + \left(\frac{-2}{2!}x^2\right) + 0x^3 + \left(\frac{8}{4!}x^3\right) = 1 - x^2 + \frac{x^4}{3}$$

Alternatives for #5

$$(\cos x)^2 = \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots\right)$$

$$= 1 + \left[(1)\left(-\frac{x^2}{2}\right) + \left(-\frac{x^2}{2}\right)(1)\right] + \left[(1)\frac{x^4}{24} + \left(-\frac{x^2}{2}\right)^2 + \left(\frac{x^4}{24}\right)(1)\right] + \cdots$$

$$= 1 - x^2 + \frac{x^4}{3} + \cdots$$

$$(\cos x)^2 = \frac{1 + \cos(2x)}{2}$$

$$= \frac{1}{2}\left(1 + \left[1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \cdots\right]\right)$$

$$= 1 - x^2 + \frac{x^4}{3} + \cdots$$

Also, easier to take derivatives of

$$(\cos x)^2 = \frac{1 + \cos(2x)}{2}$$

6. (8 points)

(a) Using the Taylor polynomial of order 2 generated by the function $f(x) = \ln(x)$ at $x = 1$, estimate the value of $\ln(\frac{1}{3})$.

$f^{(0)}(x) = \ln(x)$

$f^{(1)}(x) = 1/x$

$f^{(2)}(x) = -1/x^2$

$$\ln(x) \approx (x-1) - \frac{(x-1)^2}{2}$$

$$\ln\left(\frac{1}{3}\right) \approx \left(\frac{1}{3}-1\right) - \frac{(1/3-1)^2}{2}$$

$$-\frac{2}{3} - \frac{2}{9} = -\frac{8}{9}$$

(b) What is the maximum value of $|f^{(3)}(x)|$ on the interval $[\frac{1}{3}, 1]$?

$f^{(3)}(x) = 2/x^3$ (decreasing on $[1/3, 1]$)

Thus $M = f^{(3)}(\frac{1}{3}) = 2 \times 3^3 = 54$

(c) Find an upper bound on the absolute value of the error for the estimate from (a) using the Remainder Estimation Theorem.

$$|R_2| \leq \frac{M\left|\frac{1}{3}-1\right|^3}{3!} = \frac{54\left(\frac{2}{3}\right)^3}{6} = \frac{16}{6} = \frac{8}{3}$$

The End