University of South Carolina

Midterm Examination 2 October 23, 2018

Math 142–H01

Closed book examination Time: 75 minutes

Name Solutions

Instructions:

No notes, books, or calculators are allowed. If you need more space than is provided use the back of the previous page and clearly indicate you have done so. Simplify your final answers. Full credit may not be awarded for insufficient accompanying work.

1	9	9
2	9	9
3	9	9
4	9	9
5	6	6
6	8	8
Total	50	50

1. (9 points) Find the limit of each of the following sequences or explain why the limit does not exist.

(a) $\lim_{n\to\infty} \frac{n^2-1}{3n^2+2n} = \lim_{n\to\infty} \frac{1-1/n^2}{3+2/n} = \frac{1-0}{3+0} = \frac{1}{3}$

(b) $\lim_{n\to\infty} \frac{4(3^n)+2^{-n}}{3(2^n)-3^n} = \lim_{n\to\infty} \frac{4+2^{-n}/3^n}{3(2^n)/3^n-1}$

$= \lim_{n\to\infty} \frac{4+\left(\frac{1}{6}\right)^n}{3\left(2/3\right)^n-1} = \frac{4+0}{0-1} = -4$

(c) $\lim_{n\to\infty} \left(\frac{n-1}{n+1}\right)^n =: L$

$\ln L = \lim_{n\to\infty} n\left[\ln\left(\frac{n-1}{n+1}\right)\right] = \text{“}\infty \times 0\text{”}$ [l'Hôpital]

$= \lim_{n\to\infty} \frac{\ln(n-1)-\ln(n+1)}{1/n} = \lim_{n\to\infty} \frac{\frac{1}{n-1}-\frac{1}{n+1}}{-1/n^2}$

$= \lim_{n\to\infty} -\frac{2n^2}{n^2-1} = -2$

$L = e^{-2}$

2. (9 points) Find the value of each of the following series or explain why the series diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{3}}}$ Diverges since p-series with $p<1$.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n+3}{4^n} = \sum_{n=0}^{\infty}\left(\frac{2}{4}\right)^n + 3\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n$

$= \left(\frac{1}{1-1/2}\right) + 3\left(\frac{1}{1-1/4}\right)$

$= 2 + 3\left(\frac{4}{3}\right) = 6$

(c) $\displaystyle\sum_{n=3}^{\infty}\left(\frac{1}{3}\right)^n = \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n - \left(1+\frac{1}{3}+\frac{1}{3^2}\right)$

$= \left(\frac{1}{1-1/3}\right) - \frac{13}{9} = \frac{3}{2} - \frac{13}{9} = \frac{1}{18}$

3. (9 points) For each series, what can you conclude from the given convergence test?

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n+1}{n^2+1}$ using the Alternating Series Test.

Alternating ✓

Non-increasing ✓

$\displaystyle\lim_{n\to\infty}\frac{n+1}{n^2+1}=0$ ✓

Converges

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ using the nth Term Divergence Test.

$\displaystyle\lim_{n\to\infty}\frac{1}{n}=0$ inconclusive

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{\ln(n)}{n}$ using the Integral Test.

$$\int_1^{\infty}\frac{\ln(n)}{n} = \lim_{b\to\infty}\int_1^{b}\frac{\ln(n)}{n}\,dn = \lim_{b\to\infty}\int_0^{\ln(b)} u\,du = \infty$$

diverges

4. (9 points) For each series, what can you conclude from the given convergence test?

(a) $\sum_{n=1}^{\infty} \frac{n+1}{n-2}$ using the Ratio Test.

$$\lim_{n\to\infty} \left| \frac{\left(\frac{n+2}{n-1}\right)}{\left(\frac{n+1}{n-2}\right)} \right| = \lim_{n\to\infty} \frac{n^2-4}{n^2-1} = 1$$

inconclusive

(b) $\sum_{n=1}^{\infty} \frac{3^n n!}{(2n+1)!}$ using the Ratio Test.

$$\lim_{n\to\infty} \left| \frac{3^{n+1}(n+1)!}{(2(n+1)+1)!} \Big/ \frac{3^n n!}{(2n+1)!} \right| = \lim_{n\to\infty} \frac{3(n+1)}{(2n+3)(2n+2)} = 0$$

converges

(c) $\sum_{n=1}^{\infty} \frac{2^n}{n^n}$ using the Root Test.

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{2^n}{n^n}\right|} = \lim_{n\to\infty} \frac{2}{n} = 0$$

converges

5. (6 points)

(a) Find the sum of the first 4 terms of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

$$-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} = \frac{-12+6-4+3}{12} = -\frac{7}{12}$$

(b) Using the Alternating Series Estimation Theorem, find an upper bound on the error of the partial sum from part (a).

$$\left| -\frac{1}{5} \right| = \frac{1}{5}$$

(c) Find the sum of the first 4 terms of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} = \frac{144+36+16+9}{144} = \frac{205}{144}$$

(d) Using the bounds for the remainder in the integral test, find an upper bound on the error of the partial sum from part (c).

$$\int_4^\infty \frac{1}{n^2}\,dn = \lim_{b\to\infty} \int_4^b \frac{1}{n^2}\,dn = \lim_{b\to\infty} \left[-\frac{1}{b} - \left(-\frac{1}{4}\right) \right] = \frac{1}{4}$$

6. (8 points)

(a) Estimate the integral $\int_2^4 \frac{1}{x-1}\,dx$ using the Trapezoid rule with 4 equal subintervals.

$$\frac{1/2}{2}\left[\left(\frac{1}{2-1}\right)+2\left(\frac{1}{5/2-1}\right)+2\left(\frac{1}{3-1}\right)+2\left(\frac{1}{7/2-1}\right)+\frac{1}{(4-1)}\right]$$

$$=\frac{1}{4}\left[1+2\frac{2}{3}+2\frac{1}{2}+2\frac{2}{5}+\frac{1}{3}\right]$$

$$=\frac{1}{4}\left[\frac{15+20+15+12+5}{15}\right]=\frac{67}{60}$$

(b) Let E_T be the error of Trapezoid rule applied to the integral $\int_a^b f(x)\,dx$ with n equal subintervals. Recall that $|E_T| \le \frac{M(b-a)^3}{12n^2}$ where M is an upper bound for the values of $|f^{(2)}|$ on $[a,b]$. What is a bound on the error in the estimate from part (a).

$$f^{(1)}(x) = -\frac{1}{(x-1)^2}$$

$$f^{(2)}(x) = +\frac{2}{(x-1)^3}$$

decreasing

$$M = |f^{(2)}(2)| = 2$$

$$|E_T| \le \frac{2(4-2)^3}{12(4)^2} = \frac{1}{12}$$

(c) Using Trapezoid rule for the integral from part (a), how many equal subintervals are required to guarantee that the estimate has an error less than 0.01?

$$\frac{2(4-2)^3}{12n^2} \le 0.01$$

$$1600 \le 12n^2$$

$$\frac{400}{3} \le n^2$$

$$133.\bar{3} \le n^2$$

$$121 < n^2 < 144 \Rightarrow n = 12$$

The End