University of South Carolina

Midterm Examination 2 October 24, 2016

Math 142–005/006

Closed book examination Time: 75 minutes

Name Solutions

Instructions:

No notes, books, or calculators are allowed. If you need more space than is provided use the back of the previous page and clearly indicate you have done so. Simplify your final answers. **Full credit may not be awarded for insufficient accompanying work.**

1	9	9
2	9	9
3	9	9
4	9	9
5	12	12
6	8	8
Total	56	56

1. (9 points) Find the limit of each of the following sequences or explain why the limit does not exist.

(a) $\lim_{n\to\infty} \frac{n^3 - 2n + 1}{n^3 + n - 1} = \lim_{n\to\infty} \frac{1 - 2/n^2 + 1/n^3}{1 - 1/n^2 - 1/n^3} = 1$

Comment: Looking at the leading terms is not sufficient explanation.

(b) $\lim_{n\to\infty} \frac{n^3}{2^n} = \text{"}\frac{\infty}{\infty}\text{"}$ l'Hôpital : $= \lim_{n\to\infty} \frac{3n^2}{\ln(2)\,2^n} = \text{"}\frac{\infty}{\infty}\text{"}$

l'Hôpital: $= \lim_{n\to\infty} \frac{6n}{\ln(2)^2\,2^n} = \text{"}\frac{\infty}{\infty}\text{"}$

l'Hôpital: $= \lim_{n\to\infty} \frac{6}{\ln(2)^3\,2^n} = 0$

Comment: "bottom grows faster than top" is not sufficient explanation

(c) $\lim_{n\to\infty} (7n)^{2/n} = \lim_{n\to\infty} (7^2)^{1/n} (n^{1/n})^2 = 1 \times 1^2 = 1$

(uses rule $\lim_{n\to\infty} n^{1/n} = 1$)

Comment: It is not true that "$\infty^0 = 1$"!!!

For example: $\lim_{n\to\infty} (3^n)^{1/n} = 3$

2. (9 points) Find the value of each of the following series or explain why the series diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges: p-series with $p = \frac{1}{2} < 1$.

(b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ Since $\left|\frac{1}{3}\right| < 1$, this is a convergent geometric series. Starting at $n=1$, the geometric series formula gives:

$$= \frac{1/3}{1 - 1/3} = \frac{1}{2}$$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n - 5}{3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n - 5 \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$

Both are convergent geometric series, so:

$$= \frac{1}{1 - 2/3} - 5\left(\frac{1}{1 - 1/3}\right)$$

$$= 3 - 5\left(\frac{3}{2}\right) = -\frac{9}{2}$$

3. (9 points) For each series, what can you conclude from the given convergence test?

(a) $\sum_{n=1}^{\infty} \frac{1}{n^4}$ using the Integral Test.

$$\int_1^{\infty} \frac{1}{x^4}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x^4}\,dx = \lim_{b\to\infty} \left[\frac{-1}{3}x^{-3}\right]_1^b$$

$$= \lim_{b\to\infty} \left[-\frac{1}{3b^3} - \left(-\frac{1}{3}\right)\right] = \frac{1}{3} \text{ exists}$$

Therefore it <u>converges</u>.

(b) $\sum_{n=1}^{\infty} \frac{4^n}{n!}$ using the Ratio Test.

$$\lim_{n\to\infty} \frac{\left(\frac{4^{n+1}}{(n+1)!}\right)}{\left(4^n/n!\right)} = \lim_{n\to\infty} \frac{4^{n+1}}{4^n} \times \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} \frac{4}{n+1} = 0 < 1$$ Thus it <u>converges</u>.

(c) $\sum_{n=1}^{\infty} \frac{2^n}{n^4}$ using the Root Test.

$$\lim_{n\to\infty} \sqrt[n]{2^n/n^4} = \lim_{n\to\infty} \frac{2}{(n^{1/n})^4} = \frac{2}{1^4} = 2 > 1$$

Thus it <u>diverges</u>.

$\boxed{\text{Comment: } \lim_{n\to\infty} n^{1/n} = 1 \text{ but } \underline{\text{not}} \text{ because } (\text{something})^0 = 1}$

4. (9 points) For each series, what can you conclude from the given convergence test?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^2+1}$ using the Limit Comparison Test with $\displaystyle\sum \frac{1}{n^2}$. $\left(\sum \frac{1}{n^2} \text{ converges}\right)$

$$\lim_{n\to\infty} \frac{\left(\frac{3}{n^2+1}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{3n^2}{n^2+1} = 3 \text{ is a positive number.}$$

Thus $\sum \frac{3}{n^2+1}$ also converges

(b) $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n+1}$ using the Limit Comparison Test with $\displaystyle\sum \frac{1}{n^2}$. $\left(\sum \frac{1}{n^2} \text{ converges}\right)$

$$\lim_{n\to\infty} \frac{\left(\frac{1}{n+1}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{n^2}{n+1} = \infty$$

Thus it is inconclusive.

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n-1}$ using the Direct Comparison Test with $\displaystyle\sum \frac{1}{n}$.

$\sum a_n = \sum \frac{1}{n}$ diverges. Let $\sum b_n = \sum \frac{1}{n-1}$.

Since $-1 < 0$

$n-1 < n$

$\frac{1}{n} < \frac{1}{n-1}$

Thus $a_n < b_n$.

Since $0 < a_n < b_n$ and $\sum a_n$ diverges:

$\sum \frac{1}{n-1}$ diverges

5. (12 points) For each of the following series, determine if it converges or diverges.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2}{n^n}$ Root test: $\displaystyle\lim_{n\to\infty} \sqrt[n]{\frac{2}{n^n}} = \lim_{n\to\infty} \frac{2^{1/n}}{n}$

$$= \lim_{n\to\infty} \frac{1}{n} = 0 < 1$$

Thus it converges.

(b) $\displaystyle\sum_{n=3}^{\infty} \frac{n^3-2n+1}{n^4-4n+2}$ Limit comparison with $\sum \frac{1}{n}$ which diverges.

$$\lim_{n\to\infty} \frac{\left(\frac{n^3-2n+1}{n^4-4n+2}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{n^4-2n^2+n}{n^4-4n+n} = 1$$ is positive number

Thus it also diverges.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n!}$ Ratio test:

$$\lim_{n\to\infty} \left(\frac{(n+1)^3}{(n+1)!}\right) \Big/ \left(\frac{n^3}{n!}\right) = \lim_{n\to\infty} \frac{(n+1)^3}{n^3} \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^3 \frac{1}{n+1} = 0 < 1$$

Thus it converges.

6. (8 points) For each of the following series, determine if it

- converges absolutely,
- converges conditionally, or
- diverges.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$

$\sum \frac{1}{n^3}$ converges since it is a p-series with $p > 1$

Thus it converges absolutely.

Comment: AST says nothing about absolute convergence.

(b) $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$ $u_n = \frac{1}{\ln(n)}$

$\sum \frac{1}{\ln(n)}$ limit comparison with $\sum \frac{1}{n}$ which diverges.

$$\lim_{n\to\infty} \frac{\left(1/\ln(n)\right)}{1/n} = \lim_{n\to\infty} \frac{n}{\ln(n)} = \text{"}\frac{\infty}{\infty}\text{"}$$

l'Hôpital: $\lim_{n\to\infty} \frac{1}{(1/n)} = \infty$

Thus $\sum \frac{1}{\ln(n)}$ diverges

However, series is
- alternating
- $u_n = \frac{1}{\ln(n)}$ is decreasing
- $\lim_{n\to\infty} \frac{1}{\ln(n)} = 0$

Thus converges by alternating series test.

Conditionally convergent.

The End