**University of South Carolina**

Midterm Examination 2 October 24, 2016

**Math 142–003/004**

Closed book examination Time: 75 minutes

**Name** Solutions

**Instructions:**

No notes, books, or calculators are allowed. If you need more space than is provided use the back of the previous page and clearly indicate you have done so. Simplify your final answers. **Full credit may not be awarded for insufficient accompanying work.**

| | | |
|---|---|---|
| 1 | 9 | 9 |
| 2 | 9 | 9 |
| 3 | 9 | 9 |
| 4 | 9 | 9 |
| 5 | 12 | 12 |
| 6 | 8 | 8 |
| Total | 56 | 56 |

1. (9 points) Find the limit of each of the following sequences or explain why the limit does not exist.

(a) $\lim_{n\to\infty} \frac{4n^2+2n-1}{n^3+1} = \lim_{n\to\infty} \frac{4/n + 2/n^2 - 1/n^3}{1 + 1/n^3} = 0$

Comment: Looking at the leading terms is not sufficient explanation.

(b) $\lim_{n\to\infty} \frac{n^2}{2^n}$ = "$\frac{\infty}{\infty}$" indeterminate form (Apply l'Hôpital)

$= \lim_{n\to\infty} \frac{2n}{\ln(2)2^n}$ = "$\frac{\infty}{\infty}$" (Apply l'Hôpital)

$= \lim_{n\to\infty} \frac{2}{\ln(2)^2 2^n} = 0$

Comment: "bottom grows faster than top" is not sufficient explanation

(c) $\lim_{n\to\infty} (3n)^{4/n} = \left(\lim_{n\to\infty} \sqrt[n]{3^4}\right)\left(\lim_{n\to\infty} n^{1/n}\right)^4 = 1 \times 1^4 = 1$

Comment: It is not true that "$\infty^0 = 1$"!!!

For example: $\lim_{n\to\infty} (3^n)^{1/n} = 3$

2. (9 points) Find the value of each of the following series or explain why the series diverges.

(a) $\sum_{n=1}^{\infty} n$

$\lim_{n\to\infty} n = \infty$

Diverges by nth term divergence test.

(b) $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$

Since $|\frac{1}{3}|<1$, this is a convergent geometric series.

Starting at $n=1$, the geometric series formula gives:

$$= \frac{1/3}{1-1/3} = \frac{1}{2}$$

(c) $\sum_{n=0}^{\infty} \frac{3^n - 2}{4^n}$ $= \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n - 2 \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n$

Both are convergent geometric series, so:

$$= \frac{1}{1-3/4} - 2\,\frac{1}{1-1/4}$$

$$= 4 - 2\left(\frac{4}{3}\right) = \frac{4}{3}$$

3. (9 points) For each series, what can you conclude from the given convergence test?

(a) $\sum_{n=1}^{\infty} e^{-n}$ using the Integral Test.

$$\int_1^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_1^b e^{-x}\,dx$$

$$= \lim_{b\to\infty}\left[-e^{-x}\right]_1^b = \lim_{b\to\infty}\left(-e^{-b}+e^{-1}\right) = \frac{1}{e} \text{ exists}$$

Thus <u>converges</u>.

(b) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ using the Ratio Test.

$$\lim_{n\to\infty}\frac{\left|\frac{2^{n+1}}{(n+1)!}\right|}{\left|\frac{2^n}{n!}\right|} = \lim_{n\to\infty}\frac{2^{n+1}}{2^n}\,\frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty}\frac{2}{n+1} = 0 < 1 \quad \text{Thus } \underline{\text{converges}}.$$

(c) $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$ using the Root Test.

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{3^n}{n^3}\right|} = \lim_{n\to\infty}\frac{3}{n^{3/n}} = \frac{3}{\left(\lim_{n\to\infty} n^{1/n}\right)^3} = 3 > 1$$

Thus <u>diverges</u>.

$$\boxed{\text{Comment: } \lim_{n\to\infty} n^{1/n} = 1 \text{ but } \underline{\text{not}} \text{ because } (\text{something})^0 = 1}$$

4. (9 points) For each series, what can you conclude from the given convergence test?

(a) $\sum_{n=1}^{\infty} \frac{3}{n+1}$ using the Limit Comparison Test with $\sum \frac{1}{n}$. Note $\sum \frac{1}{n}$ diverges.

$$\lim_{n\to\infty} \frac{\left(3/n+1\right)}{\left(1/n\right)} = \lim_{n\to\infty} \frac{3n}{n+1} = 3$$ is a positive number

Thus it also diverges

(b) $\sum_{n=4}^{\infty} \frac{1}{n^2+1}$ using the Limit Comparison Test with $\sum \frac{1}{n}$. Note $\sum \frac{1}{n}$ diverges

$$\lim_{n\to\infty} \frac{\left(1/n^2+1\right)}{\left(1/n\right)} = \lim_{n\to\infty} \frac{n}{n^2+1} = 0$$

Thus it is inconclusive.

(c) $\sum_{n=2}^{\infty} \frac{1}{n-1}$ using the Direct Comparison Test with $\sum \frac{1}{n}$.

$\sum a_n = \sum \frac{1}{n}$ diverges. Let $\sum b_n = \sum \frac{1}{n-1}$.

Since $-1 < 0$

$n-1 < n$

$\frac{1}{n} < \frac{1}{n-1}$

Thus $a_n < b_n$.

Since $0 < a_n < b_n$ and $\sum a_n$ diverges:

$\sum \frac{1}{n-1}$ diverges

5. (12 points) For each of the following series, determine if it converges or diverges.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2}{(n+1)^n}$ Root test

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{2}{(n+1)^n}\right|} = \lim_{n\to\infty} \frac{2^{1/n}}{(n+1)} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

Thus it converges.

(b) $\displaystyle\sum_{n=3}^{\infty} \frac{n^2+2n-1}{n^4-2n+3}$ Limit comparison with $\sum \frac{1}{n^2}$ which converges.

$$\lim_{n\to\infty} \frac{\left(\frac{n^2+2n-1}{n^4-2n+3}\right)}{\left(1/n^2\right)} = \lim_{n\to\infty} \frac{n^4+2n^3-n^2}{n^4-2n+3} = 1$$

exists and $>0$

Thus series also converges.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(2n)!}$ Ratio test

$$\lim_{n\to\infty} \frac{(n+1)^2}{(2(n+1))!} \Big/ \frac{n^2}{(2n)!} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \frac{(2n)!}{(2n+2)!}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 \frac{1}{(2n+2)(2n+1)} = 0 < 1$$

Thus it converges.

6. (8 points) For each of the following series, determine if it

- converges absolutely,
- converges conditionally, or
- diverges.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges (Harmonic series)}$$

$u_n = \frac{1}{n}$

Original series is alternating,
- $u_n$ decreasing,
- $\lim_{n\to\infty} \frac{1}{n} = 0$

$\Rightarrow$ Converges by alternating series test.

Series is <u>conditionally convergent</u>.

(b) $\sum_{n=2}^{\infty} \frac{(-1)^{n+1} n}{\ln(n)}$

$u_n = \frac{n}{\ln(n)}$

$$\lim_{n\to\infty} \frac{n}{\ln(n)} = \text{“}\frac{\infty}{\infty}\text{''} \quad \text{(l'Hôpital)}$$

$$= \lim_{n\to\infty} \frac{1}{(1/n)} = \lim_{n\to\infty} n = \infty$$

Thus $\lim_{n\to\infty} \frac{(-1)^{n+1} n}{\ln(n)}$ does not exist.

Series <u>diverges</u> by nth term divergence test.

Comment: The alternating series test can only show <u>convergence</u>.

**The End**