University of South Carolina

Midterm Examination 2 October 20, 2016

Math 142 Section H03

Closed book examination Time: 75 minutes

Name Solutions

Instructions:

No notes, books, or calculators are allowed. If you need more space than is provided use the back of the previous page and clearly indicate you have done so. Simplify your final answers. **Full credit may not be awarded for insufficient accompanying work.**

1	12	12
2	12	12
3	12	12
4	12	12
5	12	12
6	12	12
Total	72	72

1. (12 points) Find the limit of each of the following sequences or explain why the limit does not exist.

(a) $\lim_{n\to\infty} \frac{3n^2+n}{4n^2-2} = \lim_{n\to\infty} \frac{3+\frac{1}{n}}{4-\frac{2}{n^2}} = \frac{3}{4}$

converges to $\frac{3}{4}$

(b) $\lim_{n\to\infty} \frac{\ln(n)}{n^2} =$ "$\frac{\infty}{\infty}$" l'Hôpital: $= \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{2n} = \lim_{n\to\infty} \frac{1}{2n^2} = 0$

converges to 0

(c) $\lim_{n\to\infty} (2n)^{3/n} = \lim_{n\to\infty} (2^3)^{1/n} (n^{1/n})^3 = 1 \cdot 1^3 = 1$

converges to 1

2. (12 points) Find the value of each of the following series or explain why the series diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ p-series with $p = \frac{1}{2}$

diverges

(b) $\displaystyle\sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n - \left(1 + \frac{1}{2}\right)$

$$= \frac{1}{1-\frac{1}{2}} - \frac{3}{2} = \frac{1}{2}$$

converges to $\frac{1}{2}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n + 4}{3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n + 4 \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$

$$= \frac{1}{1-\frac{2}{3}} + 4\,\frac{1}{1-\frac{1}{3}} = 3 + 4\,\frac{3}{2} = 9$$

converges to 9

3. (12 points) For each series, what can you conclude from the given convergence test?

(a) $\sum_{n=1}^{\infty} \frac{2}{n^3}$ using the Integral Test.

$$\int_1^\infty \frac{2}{x^3}\,dx = \lim_{b\to\infty}\left[-x^{-2}\right]_1^b = 0-1$$

$\Rightarrow$ converges

(b) $\sum_{n=1}^{\infty} \frac{n^2}{n!}$ using the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{\left[\frac{(n+1)^2}{(n+1)!}\right]}{\left[\frac{n^2}{n!}\right]}\right| = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2\frac{1}{n+1}$$

$$= \lim_{n\to\infty}\frac{n+1}{n^2} = 0$$

$\Rightarrow$ converges

(c) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ using the Root Test.

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{n^2}{2^n}\right|} = \lim_{n\to\infty}\frac{n^{2/n}}{2} = \frac{1}{2}$$

$\Rightarrow$ converges

4. (12 points) For each series, what can you conclude from the given convergence test?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ using the Limit Comparison Test with $\sum \frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{\left(\frac{1}{n^2+4}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{n^2}{n^2+4} = 1$$

Since $\sum \frac{1}{n^2}$ converges $\Rightarrow$ converges

(b) $\displaystyle\sum_{n=4}^{\infty} \sqrt{\frac{n-1}{n^3+1}}$ using the Limit Comparison Test with $\sum \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{\sqrt{\frac{n-1}{n^3+1}}}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \sqrt{\frac{n^3-n}{n^3+1}} = \sqrt{\lim_{n\to\infty} \frac{1-\frac{1}{n^2}}{1+\frac{1}{n^3}}} = 1$$

Since $\sum \frac{1}{n}$ diverges $\Rightarrow$ diverges

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ using the Direct Comparison Test with $\sum \frac{1}{n^2}$.

$$0 < 4 \Rightarrow n^2 < n^2+4 \Rightarrow \frac{1}{n^2+4} < \frac{1}{n^2}$$

Since $\sum \frac{1}{n^2}$ converges $\Rightarrow$ converges

5. (12 points) For each of the following series, determine if it converges or diverges.

(a) $\sum_{n=0}^{\infty} \frac{3^n}{n!}$

Ratio test: $\lim_{n\to\infty} \frac{\left[\frac{3^{n+1}}{(n+1)!}\right]}{\left[\frac{3^n}{n!}\right]} = \lim_{n\to\infty} \frac{3}{n+1} = 0$

$\Rightarrow$ converges

(b) $\sum_{n=3}^{\infty} \frac{n+1}{n^2-2}$

Limit comparison with $\sum \frac{1}{n}$:

$\lim_{n\to\infty} \left| \left(\frac{n+1}{n^2-2}\right) \Big/ \left(\frac{1}{n}\right) \right| = \lim_{n\to\infty} \frac{n^2+n}{n^2-2} = 1$

Since $\sum \frac{1}{n}$ diverges $\Rightarrow$ diverges

(c) $\sum_{n=1}^{\infty} \frac{\ln(n)}{3^{n+2}}$

Ratio test: $\lim_{n\to\infty} \left| \frac{\ln(n+1)}{3^{n+3}} \Big/ \frac{\ln(n)}{3^{n+2}} \right|$

$= \lim_{n\to\infty} \frac{1}{3} \frac{\ln(n+1)}{\ln(n)} = \text{“}\frac{\infty}{\infty}\text{”}$

l'Hôpital: $\lim_{n\to\infty} \frac{1}{3} \frac{\left(\frac{1}{n+1}\right)}{\left(\frac{1}{n}\right)} = \frac{1}{3} \lim_{n\to\infty} \frac{n}{n+1} = \frac{1}{3}$

$\Rightarrow$ converges

6. (12 points) For each of the following series, determine if it

- converges absolutely,
- converges conditionally, or
- diverges.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ Since $\sum \frac{1}{n^2}$ converges (p-series with $p=2$)

$\Rightarrow$ converges absolutely

(b) $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln(n)}$

$$\lim_{n \to \infty} \frac{1}{n \ln(n)} = 0$$

By alternating series test, converges.

Integral test: $\int_2^\infty \frac{dx}{x \ln(x)} = \lim_{b \to \infty} \int_{\ln(2)}^{\ln(b)} \frac{1}{u}\, du = \lim_{b \to \infty} \Big[\ln(u) \Big]_{\ln(2)}^{\ln(b)} = \infty$

Thus $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$ diverges

$\Rightarrow$ converges conditionally

The End