Problem 1 Find the following integrals.

(a) $\displaystyle\int 4x^3 - 2x^2 + x - 1 \; dx$

Solution: $x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2 - x + C$

(b) $\displaystyle\int e^x + 2^x + \frac{1}{x} + \sqrt{x} \; dx$

Solution: $e^x + \frac{2^x}{\ln(2)} + \ln|x| + \frac{2}{3}x^{\frac{3}{2}} + C$

(c) $\displaystyle\int \sin(\theta) + \cos(\theta) + \tan(\theta) + \sec(\theta) \; d\theta$

Solution: $-\cos(\theta) + \sin(\theta) + \ln|\sec(\theta)| + \ln|\sec(\theta) + \tan(\theta)| + C$

(d) $\displaystyle\int \frac{1}{1+t^2} + \frac{1}{\sqrt{1-t^2}} + \sec^2(t) + \sec(t)\tan(t) \; dt$

Solution: $\arctan(t) + \arcsin(t) + \tan(t) + \sec(t) + C$

Problem 2 Find the following integrals.

(a) $\displaystyle\int \frac{r^6}{4+r^7} \; dr$

Solution: Using the substitution $u = 4 + r^7$ with $du = 7r^6 \; dr$ we find:

$$\int \frac{r^6}{4+r^7} \; dr = \frac{1}{7}\int \frac{1}{u} \; du = \frac{1}{7}\ln|u| + C = \frac{1}{7}\ln|4+r^7| + C$$

(b) $\int x \sin(2x)\ dx$

Solution: We use integration by parts with $u = x$, $du = dx$, $v = -\frac{1}{2}\cos(2x)$, and $dv = \sin(2x)\ dx$. Thus:

$$\begin{aligned}\int x\sin(2x)\ dx &= (x)\left(-\frac{1}{2}\cos(2x)\right) - \int\left(-\frac{1}{2}\cos(2x)\right)\ dx \\ &= -\frac{x}{2}\cos(2x) + \frac{1}{2}\int\cos(2x)\ dx \\ &= -\frac{x}{2}\cos(2x) + \frac{1}{2}\left(\frac{1}{2}\sin(2x)\right) + C \\ &= -\frac{x}{2}\cos(2x) + \frac{1}{4}\sin(2x) + C\end{aligned}$$

(c) $\int \cos(\theta)2^{\sin(\theta)}\ d\theta$

Solution: Using the substitution $u = \sin(\theta)$ with $du = \cos(\theta)d\theta$ we find:

$$\int\cos(\theta)2^{\sin(\theta)}\ d\theta = \int 2^u\ du = \frac{2^u}{\ln(2)} + C = \frac{2^{\sin(\theta)}}{\ln(2)} + C$$

Problem 3 Find the following integrals.

(a) $\int x^2 e^x\ dx$

Solution: We apply integration by parts twice. First, with $u = x^2$, $du = 2x\ dx$, $v = e^x$ and $dv = e^x\ dx$:

$$\begin{aligned}\int x^2e^x\ dx &= (x^2)(e^x) - \int e^x 2x\ dx \\ &= x^2\cos(x) - 2\int xe^x\ dx\end{aligned}$$

then with $u = x$, $du = \ dx$, $v = e^x$ and $dv = e^x\ dx$:

$$\begin{aligned}&= x^2e^x - 2\int xe^x\ dx \\ &= x^2e^x - 2\left(xe^x - \int e^x\ dx\right) \\ &= x^2e^x - 2\left(xe^x - e^x\right) + C \\ &= (x^2 - 2x + 2)e^x + C\end{aligned}$$

(b) $\displaystyle\int \frac{u+1}{\sqrt{u-3}}\ du$

Solution: We use the substitution $t = u - 3$ with $du = dt$. Note that $u = t + 3$. Thus:

$$\begin{aligned}\int \frac{u+1}{\sqrt{u-3}}\ du &= \int \frac{(t+3)+1}{\sqrt{t}}\ dt \\ &= \int \frac{t+4}{\sqrt{t}}\ dt \\ &= \int t^{\frac{1}{2}} + 4t^{-\frac{1}{2}}\ dt \\ &= \frac{2}{3}t^{\frac{3}{2}} + 4(2)t^{\frac{1}{2}} + C \\ &= \frac{2}{3}(u-3)^{\frac{3}{2}} + 8(u-3)^{\frac{1}{2}} + C\end{aligned}$$

(c) $\displaystyle\int \tan^2(\theta)\sec^4(\theta)\ d\theta$

Solution: First, we use the identity $\sec^2(\theta) = \tan^2(\theta) + 1$ to rewrite the integrand:

$$\int \tan^2(\theta)\sec^4(\theta)\ d\theta = \int \tan^2(\theta)(\tan^2(\theta)+1)\sec^2(\theta)\ d\theta$$

Now we use the substitution $u = \tan(\theta)$ with $du = \sec^2(\theta)\ d\theta$:

$$\begin{aligned}\int \tan^2(\theta)(\tan^2(\theta)+1)\sec^2(\theta)\ d\theta &= \int u^2(1+u^2)\ du \\ &= \int u^2 + u^4\ du \\ &= \frac{1}{3}u^3 + \frac{1}{5}u^5 + C \\ &= \frac{1}{3}\tan^2(\theta) + \frac{1}{5}\tan^5(\theta) + C\end{aligned}$$

Problem 4 Find $\displaystyle\int \frac{dx}{\sqrt{4x^2+9}}$.

Solution: We build a right triangle with adjacent 3, opposite $2x$, and hypotenuse $\sqrt{4x^2+9}$. Note:

$$\tan(\theta) = \frac{2x}{3} \quad \sec^2(\theta)\ d\theta = \frac{2}{3}\ dx \quad \sec(\theta) = \frac{\sqrt{4x^2+9}}{3},$$

thus

$$x = \frac{3}{2}\tan(\theta) \quad dx = \frac{3}{2}\sec^2(\theta)\ d\theta \quad \sqrt{4x^2+9} = 3\sec(\theta).$$

We apply this substitution:

$$\begin{aligned}\int \frac{1}{\sqrt{4x^2+9}}\ dx &= \int \frac{1}{3\sec(\theta)}\frac{3}{2}\sec^2(\theta)\ d\theta \\ &= \frac{1}{2}\int \sec(\theta)\ d\theta \\ &= \frac{1}{2}\ln|\sec(\theta)+\tan(\theta)| + C \\ &= \frac{1}{2}\ln\left|\frac{\sqrt{4x^2+9}}{3} + \frac{2x}{3}\right| + C\end{aligned}$$

Problem 5 Find the following integrals.

(a) $\displaystyle\int \frac{1}{4x^2-1}\ dx$

Solution: We apply integration by parts. Since $4x^2-1 = (2x-1)(2x+1)$ we want to find A, B so that

$$\frac{1}{4x^2-1} = \frac{A}{2x-1} + \frac{B}{2x+1}$$

Putting both sides over a common denominator we want

$$1 = A(2x+1) + B(2x-1) = (2A+2B)x + (A-B).$$

Thus we obtain the system $2A+2B=0$, $A-B=1$, which has solution $A=\frac{1}{2}$, $B=-\frac{1}{2}$. Thus

$$\int \frac{1}{4x^2-1}\ dx = \frac{1}{2}\int \frac{1}{2x-1}\ dx - \frac{1}{2}\int \frac{1}{2x+1}\ dx = \frac{1}{4}\ln|2x-1| - \frac{1}{2}\ln|2x+1| + C$$

(b) $\displaystyle\int_0^1 \frac{4\ dx}{3x-2}$

Solution: This is an improper integral since the integrand has a vertical asymptote at $x=\frac{2}{3}$. Breaking it up we have

$$\begin{aligned}\int_0^1 \frac{4\,dx}{3x-2} &= \int_0^{\frac{2}{3}} \frac{4\,dx}{3x-2} + \int_{\frac{2}{3}}^1 \frac{4\,dx}{3x-2}\\ &= \left(\lim_{b\to\frac{2}{3}^-} \int_0^b \frac{4\,dx}{3x-2}\right) + \left(\lim_{a\to\frac{2}{3}^+} \int_a^1 \frac{4\,dx}{3x-2}\right)\\ &= \lim_{b\to\frac{2}{3}^-} \left[\frac{4}{3}\ln|3x-2|\right]_0^b + \lim_{a\to\frac{2}{3}^+} \left[\frac{4}{3}\ln|3x-2|\right]_a^1\\ &= \lim_{b\to\frac{2}{3}^-} \frac{4}{3}\ln|3b-2| - \frac{4}{3}\ln|3(0)-2| + \frac{4}{3}\ln|3(1)-2| - \lim_{a\to\frac{2}{3}^+} \frac{4}{3}\ln|3a-2|.\end{aligned}$$

Since $\lim_{b\to\frac{2}{3}^-} \frac{4}{3}\ln|3b-2|$ does not exist, the integral diverges.

Problem 6 Determine a value for each of the following or, if they do not have values, then show that they diverge or do not exist.

(a) $\displaystyle\lim_{n\to\infty} \frac{3-4n^2}{3n^2+2n-1}$

Solution:

$$\lim_{n\to\infty} \frac{3-4n^2}{3n^2+2n-1} \times \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{\frac{3}{n^3}-4}{3+\frac{2}{n^2}-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{0-4}{3+0-0} = -\frac{4}{3}$$

(b) $\displaystyle\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right)$

Solution: Note that $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \sin(0) = 0$. Thus we have an indeterminate form $\infty \times 0$. We need to rearrange so that l'Hôpital's rule applies.

$$\lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}$$

evaluates to the indeterminate form $0/0$, which is a quotient so we can apply l'Hôpital's rule. Our limit becomes

$$\lim_{n\to\infty} \frac{\cos\left(\frac{1}{n}\right)\left(-\frac{1}{n^2}\right)}{-\frac{1}{n^2}} = \lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos(0) = 1$$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2-2^n}{3^n}$

Solution: Applying the rules for sigma notation we obtain

$$\sum_{n=1}^{\infty} \frac{2-2^n}{3^n} = 2\left(\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n\right) - \left(\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n\right).$$

The geometric series formula *starting at* $n = 1$ is

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$$

provided that $|r| < 1$. Observe that $\frac{1}{3}$ and $\frac{2}{3}$ both have absolute values less than 1 so these are convergent geometric series. Thus our series is

$$\sum_{n=1}^{\infty} \frac{2-3^n}{5^n} = 2\left(\frac{\left(\frac{1}{3}\right)}{1-\frac{1}{3}}\right) - \frac{\left(\frac{2}{3}\right)}{1-\frac{2}{3}} = 1 - 2 = -1.$$