Problem 1 Find the following integrals.

(a) $\int 4x^3 - 2x^2 + x - 1 \; dx$

Solution: $x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2 - x + C$

(b) $\int e^x + 2^x + \frac{1}{x} + \sqrt{x} \; dx$

Solution: $e^x + \frac{2^x}{\ln(2)} + \ln|x| + \frac{2}{3}x^{\frac{3}{2}} + C$

(c) $\int \sin(\theta) + \cos(\theta) + \tan(\theta) + \sec(\theta) \; d\theta$

Solution: $-\cos(\theta) + \sin(\theta) + \ln|\sec(\theta)| + \ln|\sec(\theta) + \tan(\theta)| + C$

(d) $\int \frac{1}{1+t^2} + \frac{1}{\sqrt{1-t^2}} + \sec^2(t) + \sec(t)\tan(t) \; dt$

Solution: $\arctan(t) + \arcsin(t) + \tan(t) + \sec(t) + C$

Problem 2 Find the following integrals.

(a) $\int r\sqrt{1-r^2} \; dr$

Solution: Using the substitution $u = 1 - r^2$ with $du = -2r \; dr$ we find:

$$\int r\sqrt{1-r^2} \; dr = -\frac{1}{2}\int \sqrt{u} \; du = -\frac{1}{2}\left(\frac{2}{3}u^{\frac{3}{2}}\right) + C = -\frac{1}{3}\left(1-r^2\right)^{\frac{3}{2}} + C$$

(b) $\int xe^{4x} \; dx$

Solution: We use integration by parts with $u = x$, $du = dx$, $v = \frac{1}{4}e^{4x}$, and $dv = e^{4x}\ dx$. Thus:

$$\int xe^{4x}\ dx = (x)\left(\frac{1}{4}e^{4x}\right) - \int \frac{1}{4}e^{4x}\ dx = \frac{x}{4}e^{4x} - \frac{1}{4}\left(\frac{1}{4}e^{4x}\right) + C = \left(\frac{x}{4} - \frac{1}{16}\right)e^{4x} + C$$

(c) $\int \sqrt{\sin(\theta)}\cos(\theta)\ d\theta$

Solution: Using the substitution $u = \sin(\theta)$ with $du = \cos(\theta)d\theta$ we find:

$$\int \sqrt{\sin(\theta)}\cos(\theta)\ d\theta = \int \sqrt{u}\ du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}\left(\sin(\theta)\right)^{\frac{3}{2}} + C$$

Problem 3 Find the following integrals.

(a) $\int x^2\sin(x)\ dx$

Solution: We apply integration by parts twice. First, with $u = x^2$, $du = 2x\ dx$, $v = -\cos(x)$ and $dv = \sin(x)\ dx$:

$$\begin{aligned}\int x^2\sin(x)\ dx &= (x^2)(-\cos(x)) - \int(-\cos(x))2x\ dx\\ &= -x^2\cos(x) + 2\int x\cos(x)\ dx\end{aligned}$$

then with $u = x$, $du = dx$, $v = \sin(x)$ and $dv = \cos(x)\ dx$:

$$\begin{aligned}&= -x^2\cos(x) + 2\int x\cos(x)\ dx\\ &= -x^2\cos(x) + 2\left(x\sin(x) - \int \sin(x)\ dx\right)\\ &= -x^2\cos(x) + 2\left(x\sin(x) + \cos(x)\right) + C\\ &= -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C\end{aligned}$$

(b) $\int \frac{1}{e^x + e^{-x}}\ dx$

Solution: We use the substitution $u = e^x$ with $du = e^x\ dx$. Note that $dx = \frac{du}{e^x} = \frac{du}{u}$. Thus:

$$\int \frac{1}{e^x + e^{-x}}\ dx = \int\left(\frac{1}{u + u^{-1}}\right)\frac{1}{u}\ du = \int\left(\frac{1}{u^2+1}\right)\ du = \arctan(u) + C = \arctan(e^x) + C$$

(c) $\displaystyle\int \cos^2(4t)\ dt$

Solution: We recall the half-angle identity $\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta))$. Thus,

$$\begin{aligned}\int \cos^2(4t)\ dt &= \int \frac{1}{2}\left[1 + \cos\left(2(4t)\right)\right]\ dt \\ &= \frac{1}{2}\left(\int\ dt + \int \cos(8t)\ dt\right) \\ &= \frac{1}{2}\left(t + \frac{1}{8}\sin(8t)\right) + C \\ &= \frac{t}{2} + \frac{1}{16}\sin(8t) + C\end{aligned}$$

Problem 4 Find $\displaystyle\int \frac{dx}{\sqrt{9x^2+4}}$.

Solution: We build a right triangle with adjacent 2, opposite $3x$, and hypotenuse $\sqrt{9x^2+4}$. Note:

$$\tan(\theta) = \frac{3x}{2} \quad \sec^2(\theta)\ d\theta = \frac{3}{2}\ dx \quad \sec(\theta) = \frac{\sqrt{9x^2+4}}{2},$$

thus

$$x = \frac{2}{3}\tan(\theta) \quad dx = \frac{2}{3}\sec^2(\theta)\ d\theta \quad \sqrt{9x^2+4} = 2\sec(\theta).$$

We apply this substitution:

$$\begin{aligned}\int \frac{1}{\sqrt{9x^2+4}}\ dx &= \int \frac{1}{2\sec(\theta)}\frac{2}{3}\sec^2(\theta)\ d\theta \\ &= \frac{1}{3}\int \sec(\theta)\ d\theta \\ &= \frac{1}{3}\ln|\sec(\theta) + \tan(\theta)| + C \\ &= \frac{1}{3}\ln\left|\frac{\sqrt{9x^2+4}}{2} + \frac{3x}{2}\right| + C\end{aligned}$$

Problem 5 Find the following integrals.

(a) $\displaystyle\int \frac{2}{x^2-16}\ dx$

Solution: We apply integration by parts. Since $x^2 - 16 = (x-4)(x+4)$ we want to find A, B so that

$$\frac{2}{x^2-16} = \frac{A}{x-4} + \frac{B}{x+4}$$

Putting both sides over a common denominator we want

$$2 = A(x+4) + B(x-4) = (A+B)x + (4A-4B).$$

Thus we obtain the system $A + B = 0$, $4A - 4B = 2$, which has solution $A = \frac{1}{4}$, $B = -\frac{1}{4}$. Thus

$$\int \frac{2}{x^2-16}\,dx = \frac{1}{4}\int \frac{1}{x-4}\,dx - \frac{1}{4}\int \frac{1}{x+4}\,dx = \frac{1}{4}\ln|x-4| - \frac{1}{4}\ln|x+4| + C$$

(b) $\displaystyle\int_0^1 \frac{4\,dx}{3x-2}$

Solution: This is an improper integral since the integrand has a vertical asymptote at $x = \frac{2}{3}$. Breaking it up we have

$$\begin{aligned}
\int_0^1 \frac{4\,dx}{3x-2} &= \int_0^{\frac{2}{3}} \frac{4\,dx}{3x-2} + \int_{\frac{2}{3}}^1 \frac{4\,dx}{3x-2} \\
&= \left(\lim_{b\to\frac{2}{3}^-} \int_0^b \frac{4\,dx}{3x-2}\right) + \left(\lim_{a\to\frac{2}{3}^+} \int_a^1 \frac{4\,dx}{3x-2}\right) \\
&= \lim_{b\to\frac{2}{3}^-} \left[\frac{4}{3}\ln|3x-2|\right]_0^b + \lim_{a\to\frac{2}{3}^+} \left[\frac{4}{3}\ln|3x-2|\right]_a^1 \\
&= \lim_{b\to\frac{2}{3}^-} \frac{4}{3}\ln|3b-2| - \frac{4}{3}\ln|3(0)-2| + \frac{4}{3}\ln|3(1)-2| - \lim_{a\to\frac{2}{3}^+} \frac{4}{3}\ln|3a-2|.
\end{aligned}$$

Since $\lim_{b\to\frac{2}{3}^-} \frac{4}{3}\ln|3b-2|$ does not exist, the integral diverges.

Problem 6 Determine a value for each of the following or, if they do not have values, then show that they diverge or do not exist.

(a) $\displaystyle\lim_{n\to\infty} \frac{n^2 - n^4 + 1}{2n^3 + n - 1}$

Solution:

$$\lim_{n\to\infty} \frac{n^2-n^4+1}{2n^3+n-1} \times \frac{\left(\frac{1}{n^3}\right)}{\left(\frac{1}{n^3}\right)} = \lim_{n\to\infty} \frac{\frac{1}{n} - n + \frac{1}{n^3}}{2 + \frac{1}{n^2} - \frac{1}{n^3}} = \lim_{n\to\infty} \frac{0 - n + 0}{2 + 0 - 0} = -\infty$$

Thus, it diverges.

(b) $\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right)$

Solution: Note that $\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \sin(0) = 0$. Thus we have an indeterminate form $\infty \times 0$. We need to rearrange so that l'Hôpital's rule applies.

$$\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}$$

evaluates to the indeterminate form $0/0$, which is a quotient so we can apply l'Hôpital's rule. Our limit becomes

$$\lim_{n\to\infty} \frac{\cos\left(\frac{1}{n}\right)\left(-\frac{1}{n^2}\right)}{-\frac{1}{n^2}} = \lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos(0) = 1$$

(c) $\sum_{n=1}^{\infty} \frac{2^n - 3^n}{5^n}$

Solution: Applying the rules for sigma notation we obtain

$$\sum_{n=1}^{\infty} \frac{2^n - 3^n}{5^n} = \left(\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n\right) - \left(\sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n\right).$$

The geometric series formula *starting at* $n = 1$ is

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$$

provided that $|r| < 1$. Observe that $\frac{2}{5}$ and $\frac{3}{5}$ both have absolute values less than 1 so these are convergent geometric series. Thus our series is

$$\sum_{n=1}^{\infty} \frac{2^n - 3^n}{5^n} = \frac{\left(\frac{2}{5}\right)}{1-\frac{2}{5}} - \frac{\left(\frac{3}{5}\right)}{1-\frac{3}{5}} = \frac{2}{3} - \frac{3}{2} = -\frac{5}{6}.$$