University of South Carolina

Midterm Examination 1 September 21, 2017

Math 142–005/006

Closed book examination Time: 75 minutes

Name Solutions

Instructions:

No notes, books, or calculators are allowed. If you need more space than is provided use the back of the previous page and clearly indicate you have done so. Simplify your final answers. Full credit may not be awarded for insufficient accompanying work.

1	16	16
2	9	9
3	9	9
4	8	8
5	8	8
6	10	10
Total	60	60

1. (16 points) Find the following integrals.

(a) $\int 3x^3 + 2x^2 - 3x + 4 \, dx$

$\frac{3}{4}x^4 + \frac{2}{3}x^3 - \frac{3}{2}x^2 + 4x + C$

(b) $\int \frac{1}{r^3} + e^r + 4^r + \sqrt{r} \, dr$

$-\frac{1}{2}r^{-2} + e^r + \frac{4^r}{\ln(4)} + \frac{2}{3}r^{3/2} + C$

(c) $\int \sin(\theta) + \cos(\theta) + \tan(\theta) + \sec(\theta) \, d\theta$

$-\cos\theta + \sin\theta + \ln|\sec\theta| + \ln|\sec\theta + \tan\theta| + C$

(d) $\int \sec^2(x) + \sec(x)\tan(x) + \frac{1}{1+x^2} + \frac{1}{\sqrt{1-x^2}} \, dx$

$\tan(x) + \sec(x) + \tan^{-1}(x) + \sin^{-1}(x) + C$

2. (9 points) Find the following integrals.

(a) $\displaystyle\int 3r\sin(r^2-1)\,dr$

$\boxed{\begin{array}{l} u = r^2-1 \\ du = 2r \end{array}}$ $\displaystyle = \frac{3}{2}\int \sin(u)\,du = -\frac{3}{2}\cos u + C$

$\displaystyle = -\frac{3}{2}\cos(r^2-1) + C$

(b) $\displaystyle\int 4x\cos(x)\,dx$

$\boxed{\begin{array}{ll} u = 4x & v = \sin x \\ du = 4dx & dv = \cos x\,dx \end{array}}$ $\displaystyle = 4x\sin x - 4\int \sin x\,dx$

$= 4x\sin x + 4\cos x + C$

(c) $\displaystyle\int \tan^3(\theta)\sec^2(\theta)\,d\theta$

$\boxed{\begin{array}{l} u = \tan\theta \\ du = \sec^2\theta\,d\theta \end{array}}$ $\displaystyle \Rightarrow \int u^3\,du = \frac{1}{4}u^4 + C$

$\displaystyle = \frac{1}{4}\tan^4\theta + C$

3. (9 points) Find the following integrals.

(a) $\displaystyle\int_0^2 \frac{2x\ dx}{x^2+1}$

$\boxed{\begin{array}{l} u = x^2+1 \\ du = 2x\,dx \end{array}}$ $= \displaystyle\int_{u=1}^{u=5} \frac{du}{u} = \ln|u|\Big|_1^5 = \ln(5)$

(b) $\displaystyle\int \frac{dx}{x^2+6x+10} = \int \frac{dx}{(x+3)^2+1}$ $\boxed{\begin{array}{l} u = x+3 \\ du = dx \end{array}}$

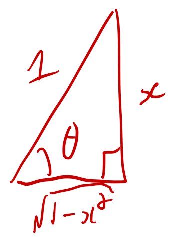
$= \displaystyle\int \frac{du}{u^2+1} = \tan^{-1}(u) + C = \tan^{-1}(x+3) + C$

(c) $\displaystyle\int x^2 e^{2x}\ dx$ $\boxed{\begin{array}{ll} u = x^2 & v = \frac{1}{2}e^{2x} \\ du = 2x\,dx & dv = e^{2x}dx \end{array}}$ $= \frac{1}{2}x^2e^{2x} - \displaystyle\int xe^{2x}\,dx$

$\boxed{\begin{array}{ll} u = x & v = \frac{1}{2}e^{2x} \\ du = dx & dv = e^{2x}dx \end{array}}$ $= \frac{1}{2}x^2e^{2x} - \left(\frac{1}{2}xe^{2x} - \displaystyle\int \frac{1}{2}e^{2x}\,dx\right)$

$= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C$

4. (8 points) Find $\int \sqrt{1-x^2}\, dx$ for $|x| < 1$. (Hint: use a trigonometric substitution)



$$\boxed{\begin{aligned} x &= \sin\theta \\ dx &= \cos\theta\, d\theta \\ \sqrt{1-x^2} &= \cos\theta \end{aligned}}$$

$$= \int \cos\theta \cos\theta\, d\theta$$

$$= \int \cos^2\theta\, d\theta = \int \frac{1}{2}\left(1+\cos 2\theta\right) d\theta$$

$$= \frac{\theta}{2} + \frac{1}{4}\sin 2\theta + C$$

$$= \frac{\sin^{-1}(x)}{2} + \frac{1}{4}\sin\left(2\sin^{-1}(x)\right) + C$$

Note: $\sin(2\theta) = 2\sin\theta\cos\theta$, thus

$\frac{\sin^{-1}(x)}{2} + \frac{1}{2}x\sqrt{1-x^2} + C$ is better answer.

5. (8 points) Find the following integrals.

(a) $\int \frac{4r-7}{r^2-3r+2}\,dr = \int \frac{A}{r-2} + \frac{B}{r-1}\,dr$

$$4r-7 = A(r-1) + B(r-2) = (A+B)r + (-A-2B) \Rightarrow \begin{array}{l} A+B=4 \\ -A-2B=-7 \end{array} \Rightarrow A=1 \quad B=3$$

$$= \int \frac{1}{r-2} + \frac{3}{r-1}\,dr = \ln|r-2| + 3\ln|r-1| + C$$

(b) $\int_0^2 \frac{1}{x-1}\,dx$ (Hint: be careful) Asymptote at $x=1$.

$$= \lim_{c\to 1^-} \int_0^c \frac{1}{x-1}\,dx + \lim_{d\to 1^+} \int_d^2 \frac{1}{x-1}\,dx$$

$$= \lim_{c\to 1^-} \Big[\ln|x-1|\Big]_0^c + \lim_{d\to 1^+} \Big[\ln|x-1|\Big]_d^2$$

$$= \lim_{c\to 1^-} \left(\ln|c-1| - 0\right) + \lim_{d\to 1^+} \left(0 - \ln|d-1|\right)$$

Limit does not exist.

$\Rightarrow$ Integral diverges.

6. (10 points)

(a) Estimate the integral $\int_1^3 \frac{dx}{x}\, dx$ using Simpson's rule with 4 equal subintervals.

$\Delta x = \frac{3-1}{4} = \frac{1}{2}$ $f(x) = 1/x$

k	x_k	$f(x_k)$
0	1	1
1	3/2	2/3
2	2	1/2
3	5/2	2/5
4	3	1/3

$$\int_1^3 \frac{dx}{x} \approx \frac{(1/2)}{3}\left(1 + 4\times\left(\frac{2}{3}\right) + 2\times\left(\frac{1}{2}\right) + 4\times\left(\frac{2}{5}\right) + \frac{1}{3}\right)$$
$$= 1.1$$

(b) Let E_S be the error of Simpson's rule applied to the integral $\int_a^b f(x)\, dx$ with n equal subintervals. Recall that $|E_S| \le \frac{M(b-a)^5}{180n^4}$ where M is an upper bound for the values of $|f^{(4)}(x)|$ on $[a, b]$. What is a bound on the error in the estimate from part (a)?

$f'(x) = -x^{-2}$

$f''(x) = 2x^{-3}$

$f^{(3)}(x) = -6x^{-4}$

$f^{(4)}(x) = 24x^{-5}$

Since f is positive and decreasing on $[1,3]$

$M = |f^{(4)}(1)| = 24$

Thus,

$$|E_S| \le \frac{24(3-1)^5}{180(4)^4} = \frac{1}{60}$$

(c) Why does part (a) give an approximation of ln(3)?

$$\int_1^3 \frac{dx}{x} = \ln(x)\Big|_1^3 = \ln(3) - \ln(1) = \ln(3).$$

The End