Problem 1 For each of the following functions:

- write down the Maclaurin series using Σ notation, and
- write down the radius of convergence.

(You do not need to justify your answers.)

(a) e^x

Solution: $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ has radius of convergence ∞

(b) $\sin(x)$

Solution: $\sin(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ has radius of convergence ∞

(c) $\cos(x)$

Solution: $\cos(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ has radius of convergence ∞

(d) $\tan^{-1}(x)$

Solution: $\tan^{-1}(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ has radius of convergence 1

Problem 2 Determine the interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{(2x-3)^n}{3n}.$$

Solution:

We apply the root test:

$$\rho = \lim_{n\to\infty} \left| \frac{(2x-3)^n}{3n} \right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{|2x-3|}{3^{\frac{1}{n}} n^{\frac{1}{n}}} = \frac{|2x-3|}{1 \cdot 1} = |2x-3|.$$

Thus the series converges when $|2x-3|<1$, diverges when $|2x-3|>1$, and the test is conclusive when $|2x-3|=1$. Solving $|2x-3|<1$ yields $-1<2x-3<1$ then $2<2x<4$ then $1<x<2$. Thus, we need to determine the convergence at $x=1$ and $x=2$.

When $x=1$, the original series becomes

$$\sum_{n=1}^{\infty}\frac{(2(1)-3)^n}{3n}=\frac{1}{3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}.$$

This is a constant multiple of the alternating harmonic series, so the series converges when $x=1$.

When $x=2$, the original series becomes

$$\sum_{n=1}^{\infty}\frac{(2(2)-3)^n}{3n}=\frac{1}{3}\sum_{n=1}^{\infty}\frac{1^n}{n}=\frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{n}.$$

This is a constant multiple of the harmonic series, so the series diverges when $x=2$.

Thus the integral of convergence is $[1,2)$.

Problem 3 The power series

$$f(x)=\sum_{n=0}^{\infty}x^n=1+x+x^2+x^3+x^4+x^5+\cdots$$
$$g(x)=\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n}=-x+\frac{x^2}{2}-\frac{x^3}{3}+\frac{x^4}{4}-\frac{x^5}{5}+\cdots$$

converge for $-1<x<1$ (you don't need to show this). Find the first 4 non-zero terms of the following power series:

(a) $f(x)+g(x)$

Solution:

$$\begin{aligned}f(x)+g(x)&=(1+x+x^2+x^3+x^4+x^5+\cdots)+(-x+\frac{x^2}{2}-\frac{x^3}{3}+\frac{x^4}{4}-\frac{x^5}{5}+\cdots)\\&=(1+0)+(1-1)x+(1+\frac{1}{2})x^2+(1-\frac{1}{3})x^3+(1+\frac{1}{4})x^4\cdots\\&=1+\frac{3}{2}x^2+\frac{2}{3}x^3+\frac{5}{4}x^4+\cdots\end{aligned}$$

(b) $f\left(\frac{x^3}{2}\right)$

Solution:

$$\begin{aligned} f\left(\frac{x^3}{2}\right) &= 1 + \left(\frac{x^3}{2}\right) + \left(\frac{x^3}{2}\right)^2 + \left(\frac{x^3}{2}\right)^3 + \left(\frac{x^3}{2}\right)^4 + \cdots \\ &= 1 + \frac{1}{2}x^3 + \frac{1}{4}x^6 + \frac{1}{8}x^9 + \cdots \end{aligned}$$

(c) $g'(x)$

Solution:

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(-x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} + \cdots\right) \\ &= -1 + \frac{2x}{2} - \frac{3x^2}{3} + \frac{4x^3}{4} - \frac{5x^4}{5} + \cdots \\ &= -1 + x - x^2 + x^3 + \cdots \end{aligned}$$

(d) $\int_0^x g(y)\ dy$

Solution:

$$\begin{aligned} \int_0^x g(y)\ dy &= \int_0^x \left(-y + \frac{y^2}{2} - \frac{y^3}{3} + \frac{y^4}{4} - \frac{y^5}{5} + \cdots\right)\ dy \\ &= \left[-\frac{y^2}{2} + \frac{y^3}{6} - \frac{y^4}{12} + \frac{y^5}{20} + \cdots\right]_0^x \\ &= -\frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{20} + \cdots \end{aligned}$$

Problem 4 Determine the Taylor polynomial of order 3 generated by the function $\sqrt{x}$ at $x = \frac{1}{4}$.

Solution: We find

n	$f^{(n)}(x)$	$f^{(n)}(\frac{1}{4})$
0	$x^{\frac{1}{2}}$	$\frac{1}{2}$
1	$\frac{1}{2}x^{-\frac{1}{2}}$	1
2	$-\frac{1}{4}x^{-\frac{3}{2}}$	-2
3	$\frac{3}{8}x^{-\frac{5}{2}}$	12

Thus, our Taylor polynomial is:

$$\begin{aligned} P_3(x) &= \sum_{n=0}^{3} \frac{f^{(n)}\left(\frac{1}{4}\right)}{n!}\left(x-\frac{1}{4}\right)^n \\ &= \frac{\left(\frac{1}{2}\right)}{1}\left(x-\frac{1}{4}\right)^0 + \frac{1}{1}\left(x-\frac{1}{4}\right)^1 + \frac{-2}{2}\left(x-\frac{1}{4}\right)^2 + \frac{12}{6}\left(x-\frac{1}{4}\right)^3 \\ &= \frac{1}{2} + \left(x-\frac{1}{4}\right) - \left(x-\frac{1}{4}\right)^2 + 2\left(x-\frac{1}{4}\right)^3 \end{aligned}$$

Problem 5 Find the following:

(a) $\lim_{x\to 0} \frac{x-\sin(x)}{x^3 e^x}$

Solution:

$$\begin{aligned} \lim_{x\to 0} \frac{x-\sin(x)}{x^3 e^x} &= \lim_{x\to 0} \frac{x - \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right)}{x^3\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)} \\ &= \lim_{x\to 0} \frac{\frac{x^3}{6} - \frac{x^5}{120} + \cdots}{x^3 + x^4 + \frac{x^5}{2} + \cdots} \\ &= \lim_{x\to 0} \left(\frac{\frac{1}{6} - \frac{x^2}{120} + \cdots}{1 + x + \frac{x^2}{2} + \cdots}\right)\frac{x^3}{x^3} \\ &= \frac{\frac{1}{6} + 0 + 0 + \cdots}{1 + 0 + 0 + \cdots} = \frac{1}{6} \end{aligned}$$

(b) $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$

Solution:

We apply the geometric series formula:

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\left(\frac{1}{2}\right)}{\left(1-\frac{1}{2}\right)} = 1$$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n}}{(2n)!}$

Solution:

Recall that $\cos(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, which converges for all x. Thus $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n}}{(2n)!} = \cos(3)$

Problem 6 Use the Taylor polynomial of order 3 generated by $\sqrt{1+x}$ at $x = 0$ to estimate

$$\int_0^1 \sqrt{1+2x^2}\ dx\ .$$

Solution:

The Taylor polynomial of order 3 generated by $\sqrt{1+x}$ at $x = 0$ is given by

$$\begin{aligned} P_3(z) &= \sum_{n=0}^{3} \binom{\frac{1}{2}}{n} z^n \\ &= \binom{\frac{1}{2}}{0} z^0 + \binom{\frac{1}{2}}{1} z^1 + \binom{\frac{1}{2}}{2} z^2 + \binom{\frac{1}{2}}{3} z^3 \\ &= 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \frac{1}{16}z^3 \end{aligned}$$

Thus

$$\begin{aligned} \int_0^1 \sqrt{1+2x^2}\ dx &\approx \int_0^1 P_3(2x^2)\ dx \\ &= \int_0^1 1 + \frac{1}{2}(2x^2) - \frac{1}{8}(2x^2)^2 + \frac{1}{16}(2x^2)^3\ dx \\ &= \int_0^1 1 + x^2 - \frac{1}{2}x^4 + \frac{1}{2}x^6\ dx \\ &= \left[x + \frac{1}{3}x^3 - \frac{1}{10}x^5 + \frac{1}{14}x^7 \right]_0^1 \\ &= 1 + \frac{1}{3} - \frac{1}{10} + \frac{1}{14} = \frac{210 + 70 - 21 + 15}{210} = \frac{274}{210} = \frac{137}{105} \end{aligned}$$