Problem 1 Find the limit of each of the following sequences or explain why the limit does not exist.

(a) $\lim_{n\to\infty} \frac{n^3 - 5n}{2n^3 + 4n}$

Solution:

$$\lim_{n\to\infty} \frac{n^3 - 5n}{2n^3 + 4n} = \lim_{n\to\infty} \frac{n^3 - 5n}{2n^3 + 4n} \times \frac{1/n^3}{1/n^3} = \lim_{n\to\infty} \frac{1 - \frac{5}{n^2}}{2 + \frac{4}{n^2}} = \frac{1-0}{2-0} = \frac{1}{2}$$

(b) $\lim_{n\to\infty} \left(\frac{1}{n}\right)^n$

Solution:

$$L = \lim_{n\to\infty} \left(\frac{1}{n}\right)^n = \lim_{n\to\infty} \frac{1}{n^n} = 0$$

Comments: Almost everyone overthought this problem! Note that 0^∞ is *not* an indeterminate form: the corresponding limit is always 0. However, you do not need to memorize this fact. At the end of the day, you can only apply l'Hôpital's rule to $\frac{0}{0}$ and $\pm\frac{\infty}{\infty}$. Let's see how the solution would have gone if you were unsure of 0^∞. Taking logarithms you find

$$\ln(L) = \lim_{n\to\infty} n \ln\left(\frac{1}{n}\right).$$

Note that $\lim_{n\to\infty} \ln(\frac{1}{n}) = -\infty$ so you should find "$\ln(L) = \infty \times (-\infty) = -\infty$"; this is also not an indeterminate form. Since $\ln(L) \to -\infty$ we conclude $L = 0$. If you insist on continuing anyway, you have

$$\ln(L) = \lim_{n\to\infty} \frac{\left(\ln(\frac{1}{n})\right)}{\left(\frac{1}{n}\right)}.$$

This gives "$\ln(L) = \frac{-\infty}{0^+} = -\infty$," which is still not an indeterminate form. At no point were we able to apply l'Hôpital's Rule.

(c) $\lim_{n\to\infty} \frac{2 - 3^n}{5 - 3^n}$

Solution:

$$\lim_{n\to\infty} \frac{2 - 3^n}{5 - 3^n} = \lim_{n\to\infty} \frac{2 - 3^n}{5 - 3^n} \times \frac{1/3^n}{1/3^n} = \lim_{n\to\infty} \frac{\frac{2}{3^n} - 1}{\frac{5}{3^n} - 1} = \frac{0-3}{0-3} = 1$$

Problem 2 Find the value of each of the following series or explain why the series diverges.

(a) $\sum_{n=1}^{\infty} n$

Solution:

Since $\lim_{n\to\infty} n = \infty$ is not zero, the series diverges by the divergence test.

(b) $\sum_{n=0}^{\infty} \frac{4-2^n}{3^n}$

Solution:

$$\sum_{n=0}^{\infty} \frac{4-2^n}{3^n} = 4\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n - \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = 4\left(\frac{1}{1-\frac{1}{3}}\right) - \left(\frac{1}{1-\frac{2}{3}}\right) = 6-3=3$$

(c) $\sum_{n=2}^{\infty} \left(\frac{1}{7}\right)^n$

Solution:

$$\sum_{n=2}^{\infty}\left(\frac{1}{7}\right)^n = \sum_{n=0}^{\infty}\left(\frac{1}{7}\right)^n - \left(\frac{1}{7}\right)^0 - \left(\frac{1}{7}\right)^1 = \frac{1}{1-\frac{1}{7}} - 1 - \frac{1}{7} = \frac{7}{6} - 1 - \frac{1}{7} = \frac{49-42-6}{42} = \frac{1}{42}$$

Problem 3 For each series, what can you conclude from the given convergence test?

(a) $\sum_{n=0}^{\infty} \frac{n!}{(2n)!}$ using the Ratio Test.

Solution:

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{(n+1)!}{(2(n+1))!}\Big/\frac{n!}{(2n)!}\right| \\ &= \lim_{n\to\infty}\left(\frac{(n+1)!}{n!}\frac{(2n)!}{(2(n+1))!}\right) \\ &= \lim_{n\to\infty}\left(\frac{(n+1)\times n!}{n!}\frac{(2n)!}{(2n+2)\times(2n+1)\times(2n)!}\right) \\ &= \lim_{n\to\infty}\left(\frac{n+1}{(2n+2)(2n+1)}\right) = 0\end{aligned}$$

Since $\rho < 1$, the series converges by Ratio Test.

(b) $\sum_{n=0}^{\infty} 2^{-n}$ using the Integral Test.

Solution:

We find the integral $\int_0^\infty 2^{-x}\,dx$ via:

$$\int_0^\infty 2^{-x}\,dx = \lim_{b\to\infty}\int_0^b 2^{-x}\,dx = \lim_{b\to\infty}\left[-\frac{1}{\ln(2)}2^{-x}\right]_0^b = \frac{1}{\ln(2)}\lim_{b\to\infty}(-2^{-b}+1) = \frac{1}{\ln(2)}$$

Since the integral converges, the series converges by integral test.

(c) $\sum_{n=1}^{\infty} \frac{4^n}{n^3}$ using the Root Test.

Solution:

$$\rho = \lim_{n\to\infty}\left|\frac{4^n}{n^3}\right|^{\frac{1}{n}} = \lim_{n\to\infty}\frac{4}{n^{\frac{3}{n}}} = \frac{4}{\left(\lim_{n\to\infty} n^{\frac{1}{n}}\right)^3} = \frac{4}{1^3} = 4$$

Since $\rho > 1$, the series diverges by Root Test.

Problem 4 For each series, what can you conclude from the given convergence test?

(a) $\sum_{n=1}^{\infty} \frac{4n}{n^2+2}$ using the Limit Comparison Test with $\sum \frac{1}{n^2}$.

Solution:

$$\lim_{n\to\infty} \frac{4n}{n^2+2}\Big/\frac{1}{n^2} = \lim_{n\to\infty} \frac{4n^3}{n^2+2} = \infty$$

Since $\sum \frac{1}{n^2}$ converges, the Limit Comparison Test is inconclusive in this case.

(b) $\sum_{n=4}^{\infty} \frac{1}{n^3+1}$ using the Limit Comparison Test with $\sum \frac{1}{n^3}$.

Solution:

$$\lim_{n\to\infty} \frac{1}{n^3+1}\Big/\frac{1}{n^3} = \lim_{n\to\infty} \frac{n^3}{n^3+2} = 1$$

Since $\sum \frac{1}{n^3}$ converges, the series converges by the Limit Comparison Test.

(c) $\sum_{n=2}^{\infty} \frac{1}{n^2+1}$ using the Direct Comparison Test with $\sum \frac{1}{n^2}$.

Solution:
Observe that $1 > 0 \implies n^2+1 > n^2 \implies \frac{1}{n^2} > \frac{1}{n^2+1}$. Since $\sum \frac{1}{n^2}$ converges, the series converges by the Direct Comparison test.

Problem 5 For each of the following series, determine if it converges or diverges.

(a) $\sum_{n=0}^{\infty} \frac{3^n}{(2n+1)!}$

Solution:
We apply the ratio test.

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{3^{n+1}}{(2(n+1)+1)!}\Big/\frac{3^n}{(2n+1)!} \right| \\
&= \lim_{n\to\infty} \left(\frac{3^{n+1}}{3^n} \frac{(2n+1)!}{(2n+3)!} \right) \\
&= \lim_{n\to\infty} \left(\frac{3}{(2n+3)\times(2n+2)} \right) = 0
\end{aligned}$$

Since $\rho < 1$, the series converges by the Ratio Test.

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{4n^2+1}{n^3-1}$

Solution:

We use the Limit Comparison Test with $\frac{1}{n}$, which diverges.

$$\lim_{n\to\infty} \frac{4n^2+1}{n^3-1} \Big/ \frac{1}{n} = \lim_{n\to\infty} \frac{4n^3+n}{n^3-1} = 4$$

Thus the series diverges by the Limit Comparison Test.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{(n+5)^n}$

Solution:

We use the Root Test.

$$\rho = \lim_{n\to\infty} \left|\frac{2^n}{(n+5)^n}\right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{2}{n+5} = 0$$

Since $\rho < 1$, the series converges by the Root Test.

Problem 6 For each of the following series, determine if it converges absolutely, converges conditionally, or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

Solution:

Observe that $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges since it is a p-series with $p = \frac{1}{2}$. Thus the series does not converge absolutely.

Now note that $a_n = (-1)^n b_n$ where $b_n = \frac{1}{\sqrt{n}}$ is positive and decreasing. We find $\lim_{n\to\infty} b_n = 0$. Thus the series converges by the alternating series test.

We conclude that the series converges conditionally.

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n!}$

Solution:

We apply the Ratio Test.

$$\begin{aligned}\rho &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(n+1)!} \Big/ \frac{(-1)^n}{n!} \right| \\ &= \lim_{n\to\infty} \frac{n!}{(n+1)!} \\ &= \lim_{n\to\infty} \frac{1}{n+1} = 0\end{aligned}$$

Thus the series converges absolutely by the Ratio Test.

Comments: Note that you do not need to use the Alternating Series Test here (although it is not wrong to do so).