Problem 1 Find the following integrals.

(a) $\int 3x^3 + 4x^2 - 2x + 7\ dx$

Solution: $\frac{3}{4}x^4 - \frac{4}{3}x^3 - x^2 + 7x + C$

(b) $\int \frac{1}{x} + e^x + 2^x + \sqrt{x}\ dx$

Solution: $\ln|x| + e^x + \frac{2^x}{\ln(2)} + \frac{2}{3}x^{\frac{3}{2}} + C$

(c) $\int \sin(\theta) + \cos(\theta) + \tan(\theta) + \sec(\theta)\ d\theta$

Solution: $-\cos(\theta) + \sin(\theta) + \ln|\sec(\theta)| + \ln|\sec(\theta) + \tan(\theta)| + C$

(d) $\int \sec^2(t) + \sec(t)\tan(t) + \frac{1}{1+t^2} + \frac{1}{\sqrt{1-t^2}}\ dt$

Solution: $\arctan(t) + \arcsin(t) + \tan(t) + \sec(t) + C$

Problem 2 Find the following integrals.

(a) $\int 3x^2\cos(x^3)\ dx$

Solution: Using the substitution $u = x^3$ with $du = 3x^2\ dx$ we find:

$$\int 3x^2\cos(x^3)\ dr = \cos(u)\ du = \sin(u) + C = \sin(x^3) + C$$

(b) $\int 4\theta\cos(3\theta)\ d\theta$

Solution: We use integration by parts with $u = 4\theta$, $du = 4d\theta$, $v = \frac{1}{3}\sin(3\theta)$, and $dv = \cos(3\theta)\ dx$. Thus:

$$\begin{aligned}\int 4\theta\cos(3\theta)\ d\theta &= (4\theta)\left(\frac{1}{3}\sin(3\theta)\right) - \int \frac{4}{3}\sin(3\theta)\ d\theta \\ &= \frac{4\theta}{3}\sin(3\theta) - \frac{4}{3}\left(-\frac{1}{3}\cos(3\theta)\right) + C \\ &= \frac{4\theta}{3}\sin(3\theta) + \frac{4}{9}\cos(3\theta) + C\end{aligned}$$

(c) $\int \tan^2(\theta)\sec^2(\theta)\ d\theta$

Solution: Using the substitution $u = \tan(\theta)$ with $du = \sec^2(\theta)d\theta$ we find:

$$\int \tan^2(\theta)\sec^2(\theta)\ d\theta = \int u^2\ du = \frac{1}{3}u^3 + C = \frac{1}{3}\tan^3(\theta) + C$$

Problem 3 Find the following integrals.

(a) $\int \frac{\ln(\ln(x))}{x}\ dx$

Solution: Using the substitution $z = \ln(x)$ with $dz = \frac{1}{z}\ dz$ we find:

$$\int \frac{\ln(\ln(x))}{x}\ dx = \int \ln(z)\ dz$$

Then, we apply integration by parts with $u = \ln(z)$, $du = \frac{1}{z}\ dz$, $v = z$ and $dv = dz$:

$$\begin{aligned}&= (\ln(z))(z) - \int z\frac{1}{z}\ dz \\ &= z\ln(z) - z + C \\ &= \ln(x)\ln(\ln(x)) - \ln(x) + C\end{aligned}$$

(b) $\int x^2 e^x\ dx$

Solution: We apply integration by parts twice. First, with $u = x^2$, $du = 2x\ dx$, $v = e^x$ and $dv = e^x\ dx$:

$$\begin{aligned}\int x^2 e^x\ dx &= (x^2)(e^x) - \int e^x 2x\ dx \\ &= x^2 e^x - 2\int xe^x\ dx\end{aligned}$$

then with $u = x$, $du = \ dx$, $v = e^x$ and $dv = e^x\ dx$:

$$\begin{aligned} &= -x^2e^x - 2\left(xe^x - \int e^x\ dx\right) \\ &= -x^2e^x - 2\left(xe^x - e^x\right) + C \\ &= (x^2 - 2x + 2)e^x + C \end{aligned}$$

(c) $\displaystyle\int \frac{1}{e^x + e^{-x}}\ dx$

Solution: We use the substitution $u = e^x$ with $du = e^x\ dx$. Note that $dx = \frac{du}{e^x} = \frac{du}{u}$. Thus:

$$\int \frac{1}{e^x + e^{-x}}\ dx = \int \left(\frac{1}{u + u^{-1}}\right)\frac{1}{u}\ du = \int \left(\frac{1}{u^2 + 1}\right)\ du = \arctan(u) + C = \arctan(e^x) + C$$

Problem 4 Find $\displaystyle\int \frac{\sqrt{9x^2 - 4}}{x}\ dx$ for $|x| > 2/3$.

Solution: We build a right triangle with adjacent 2, hypoteneuse $3x$, and opposite $\sqrt{9x^2 - 4}$. Note:

$$\sec(\theta) = \frac{3x}{2} \quad \sec(\theta)\tan(\theta)\ d\theta = \frac{3}{2}\ dx \quad \tan(\theta) = \frac{\sqrt{9x^2 - 4}}{2},$$

thus

$$x = \frac{2}{3}\sec(\theta) \quad dx = \frac{2}{3}\sec(\theta)\tan(\theta)\ d\theta \quad \sqrt{9x^2 - 4} = 2\tan(\theta).$$

We apply this substitution:

$$\begin{aligned} \int \frac{\sqrt{9x^2 - 4}}{x}\ dx &= \int \frac{2\tan(\theta)}{\frac{2}{3}\sec(\theta)}\frac{2}{3}\sec(\theta)\tan(\theta)\ d\theta \\ &= 2\int \tan^2(\theta)\ d\theta \\ &= 2\int \sec^2(\theta) - 1\ d\theta \\ &= 2\left(\tan(\theta) - \theta\right) + C \\ &= \sqrt{9x^2 - 4} - 2\,\mathrm{arcsec}\left(\frac{3x}{2}\right) + C \end{aligned}$$

Comments: Don't forget that dx needs to be replaced with $\frac{2}{3}\sec(\theta)\tan(\theta)\ d\theta$. This was the most common error.

Problem 5 Find the following integrals.

(a) $\displaystyle\int \frac{1}{x^2-3x+2}\,dx$

Solution: We find the partial fraction decomposition. Since $x^2-3x+2=(x-1)(x-2)$ we want to find A, B so that

$$\frac{1}{x^2-3x+2}=\frac{A}{x-1}+\frac{B}{x-2}$$

Putting both sides over a common denominator we want

$$1 = A(x-1)+B(x-2)=(A+B)x+(-2A-B).$$

Thus we obtain the system $A+B=0$, $-A-2B=1$, which has solution $A=-1$, $B=1$. Thus

$$\int \frac{1}{x^2-3x+2}\,dx=\int\frac{-1}{x-1}\,dx+\int\frac{1}{x-2}\,dx=-\ln|x-1|+\ln|x-2|+C$$

(b) $\displaystyle\int \frac{x}{x^2+6x+10}\,dx$

Solution: Completing the square we find $x^2+6x+10=(x+3)^2+1$. This suggests the substitution $u=x+3$. Thus

$$\begin{aligned}\int\frac{x}{x^2+6x+10}\,dx &= \int\frac{(x+3)-3}{(x+3)^2+1}\,dx\\ &= \int\frac{u-3}{u^2+1}\,du\\ &= \int\frac{u}{u^2+1}\,du-3\int\frac{1}{u^2+1}\,du\\ &= \frac{1}{2}\ln(u^2+1)-3\arctan(u)+C\\ &= \frac{1}{2}\ln(x^2+6x+10)-3\arctan(x+3)+C\end{aligned}$$

Comments: I was pleased by how many students knew to complete the square. Unfortunately, there was confusion about what to do afterwards.

Problem 6 Find the integral $\displaystyle\int_{-\infty}^{\infty}\frac{5}{9x^2+16}\,dx.$

Solution: First, let us consider the indefinite integral.

$$\begin{aligned}\int \frac{5}{9x^2+16}\,dx &= \frac{5}{9}\int \frac{1}{x^2+(\frac{4}{3})^2}\,dx \\ &= \frac{5}{9}\left(\frac{3}{4}\arctan\left(\frac{3x}{4}\right)\right)+C \\ &= \frac{5}{12}\arctan\left(\frac{3x}{4}\right)+C\end{aligned}$$

Now we return to the definite integral. We break the integral into two separate improper integrals:

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{5}{9x^2+16}\,dx &= \int_{-\infty}^{0} \frac{5}{9x^2+16}\,dx + \int_{0}^{\infty} \frac{5}{9x^2+16}\,dx \\ &= \lim_{a\to-\infty}\int_{a}^{0} \frac{5}{9x^2+16}\,dx + \lim_{b\to\infty}\int_{0}^{b} \frac{5}{9x^2+16}\,dx \\ &= \lim_{a\to-\infty}\left[\frac{5}{12}\arctan\left(\frac{3x}{4}\right)\right]_a^0 + \lim_{b\to\infty}\left[\frac{5}{12}\arctan\left(\frac{3x}{4}\right)\right]_0^b \\ &= \frac{5}{12}\arctan(0) - \frac{5}{12}\lim_{a\to-\infty}\arctan\left(\frac{3a}{4}\right) + \frac{5}{12}\lim_{b\to\infty}\arctan\left(\frac{3b}{4}\right) - \frac{5}{12}\arctan(0) \\ &= 0 - \frac{5}{12}\left(-\frac{\pi}{2}\right) + \frac{5}{12}\left(\frac{\pi}{2}\right) - 0 = \frac{5\pi}{12}\end{aligned}$$

Comments: Expressions like $\arctan(\infty)$ don't really make sense. I know what you mean, but you're going to end up making errors if you start using ∞ as a "number." An alternate approach to the integral is to do $\lim_{a\to-\infty}\lim_{b\to\infty}\int_a^b \frac{5}{9x^2+16}\,dx$ instead. This will also work, but you *cannot* use the same variable for both the lower and upper bounds: it will give you the wrong answer in many cases.