Convergence Tests

Geometric series:

$\sum_{n=0}^{\infty} r^n$ converges when $|r| < 1$ and diverges when $|r| \geq 1$.

p-series:

$\sum_{n=0}^{\infty} \frac{1}{n^p}$ converges when $p > 1$ and diverges when $p \leq 1$.

Divergence test:

If $\lim_{n\to\infty} a_n \neq 0$ or does not exist, then $\sum_{n=0}^{\infty} a_n$ diverges.

Integral test:

If f is a positive, continuous, decreasing function such that $a_n = f(n)$ for all n, then $\sum_{n=0}^{\infty} a_n$ and $\int_0^{\infty} f(x)\, dx$ either both converge or both diverge.

Alternating series test:

Suppose

- $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$,
- $\{b_n\}$ is non-increasing and positive, and
- $\lim_{n\to\infty} b_n = 0$.

Then $\sum_{n=0}^{\infty} a_n$ converges.

(Direct) Comparison test:

Suppose $0 \leq a_n \leq b_n$ for all n.

If $\sum_{n=0}^{\infty} b_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.

If $\sum_{n=0}^{\infty} a_n$ diverges, then $\sum_{n=0}^{\infty} b_n$ diverges.

Limit comparison test:

Suppose a_n, b_n are positive for all n and $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.

If $L \neq 0$ and exists, then $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ either both converge or both diverge.

If $L = 0$ and $\sum_{n=0}^{\infty} b_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.

If $L = \infty$ and $\sum_{n=0}^{\infty} b_n$ diverges, then $\sum_{n=0}^{\infty} a_n$ diverges.

Ratio test:

Suppose every a_n is non-zero and $\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$.

If $\rho < 1$, then $\sum_{n=0}^{\infty} a_n$ converges absolutely.

If $\rho > 1$ or $\rho = \infty$, then $\sum_{n=0}^{\infty} a_n$ diverges.

Root test:

Suppose $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$.

If $\rho < 1$, then $\sum_{n=0}^{\infty} a_n$ converges absolutely.

If $\rho > 1$ or $\rho = \infty$, then $\sum_{n=0}^{\infty} a_n$ diverges.