Solutions

Problem 1 Determine whether each of the following statements are true or false. No justification is necessary.

1. If H is a subgroup of a finite group G, then the order of H divides the order of G.
2. Suppose G is a finite group of order n and $x \in G$. Then $x^n = e$.
3. Every homomorphism of abelian groups is an isomorphism.
4. If G and H are subgroups of the same order, then $G \cong H$.
5. If $\varphi : G \to H$ is a group homomorphism, then $\text{im}(\varphi)$ is a subgroup of H.

Solution: 1) **True**. This is Lagrange's Theorem.
2) **True**. The order of every element of G divides the order of G.
3) **False**. Consider the zero homomorphism $f : \mathbb{Z}_2 \to \mathbb{Z}_3$ given by $f(x) = 0$ for all $x \in \mathbb{Z}_2$.
4) **False**. Consider $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_4$.
5) **True**. This follows from Theorem 12.6(i) in the textbook.

Problem 2 Let $G = \mathbb{Z}_{30} \times \mathbb{Z}_{10}$ and let $H = \langle (25, 5) \rangle$ be a normal subgroup. Determine the order of the group G/H.

Solution: The order of 25 in $\mathbb{Z}_{30}$ is $30/\gcd(25, 30) = 6$. The order of 5 in $\mathbb{Z}_{10}$ is $10/\gcd(5, 10) = 2$. The order of $(25, 5)$ in G is the lcm of these orders; thus $o(25, 5) = 6$. Therefore, H is a cyclic group of order 6. Since $|G| = |\mathbb{Z}_{30}| \times |\mathbb{Z}_{10}| = 30 \times 10 = 300$, we conclude that G/H has order $300/6 = 50$.

Problem 3 Determine the isomorphism classes of finite abelian groups of order 72.

Solution: Observe that $72 = 2^3 \times 3^2$. By the Fundamental Theorem of Abelian groups, we have the possibilities:

$$\mathbb{Z}_8 \times \mathbb{Z}_9 \qquad \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \qquad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9$$
$$\mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \qquad \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \qquad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$$

Problem 4 Let $\varphi : \mathbb{R} \to \mathrm{GL}_2(\mathbb{R})$ the function given by

$$\varphi(a) := \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

Prove that φ is a group homomorphism.

Solution: Let $a, b \in \mathbb{R}$. We observe that

$$\psi(a)\psi(b) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix} = \psi(a+b).$$

Thus φ is a group homomorphism.

Problem 5 Let $\varphi : G \to H$ be a group homomorphism. Prove that φ is injective if and only if $\ker(\varphi) = \{e\}$.

Solution: Suppose φ is injective. Then there is at most one $x \in G$ such that $\varphi(x) = e$. Since $\varphi(e) = e$, we see that $\ker(\varphi) = \{e\}$.

Conversely, suppose $\ker(\varphi) = \{e\}$. Suppose $x, y \in G$ and $\varphi(x) = \varphi(y)$. Then $\varphi(x)\varphi(y)^{-1} = e$. Thus $\varphi(xy^{-1}) = e$. Since $\ker(\varphi) = \{e\}$, we conclude that $xy^{-1} = e$. Therefore $x = y$. Thus φ is injective.