Solutions

Problem 1 Determine whether each of the following statements are true or false. No justification is necessary.

1. An element x of a group G has order n if and only if $x^n = e$.
2. If H and K are subgroups of a group G, then $H \cap K$ is a subgroup.
3. If G, H are groups, then $|G \times H| = |G| \times |H|$.
4. A product of odd permutations is odd.
5. Let H be a subgroup of G. Every right coset of H in G is also a subgroup of G.

Solution: 1) **False**. $x = (12)$ satisfies $x^4 = e$, but has order 2.
2) **True**. This is Theorem 5.4 from the text.
3) **True**. This is true for all sets. In particular, it is true for groups.
4) **False**. (12) is odd, but $(12)(12) = e$ is even.
5) **False**. $1 + 2\mathbb{Z}$ is a right coset of $H = 2\mathbb{Z}$ in $G = \mathbb{Z}$, but is not a subgroup since it does not contain the identity.

Problem 2 Determine the order of $(6, 70)$ in $\mathbb{Z}_{60} \times \mathbb{Z}_{80}$.

Solution: Using Theorem 4.4(iii), we see that $o(6) = 60/\gcd(6, 60) = 10$ in $\mathbb{Z}_{60}$, while $o(70) = 80/\gcd(70, 80) = 8$ in $\mathbb{Z}_{80}$. Using Theorem 6.1(i), we see that $o(6, 70) = \operatorname{lcm}(10, 8) = 40$ in $\mathbb{Z}_{60} \times \mathbb{Z}_{80}$.

Problem 3 Rewrite the permutation

$$(1234)(56)(142)^{-1}(14)(15)$$

as a product of disjoint cycles.

Solution: Note that $(142)^{-1} = (241)$. We trace $\{1, \ldots, 6\}$ to find the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 4 & 3 & 2 & 5 \end{pmatrix}.$$

Reading off the cycles, we obtain $(1652)(34)$.

Problem 4 Let g be an element of a group G. Let H be the subset of G such that $hg = gh$ for all $h \in H$. Prove that H is a subgroup of G.

Solution: Observe that $eg = ge$, so $e \in H$. Thus H is nonempty. Let $a, b \in H$. Thus $ag = ga$ and $bg = gb$, and we conclude that $abg = agb = gab$. Thus $ab \in H$ and H is closed under multiplication. Given $a \in H$, we know that $ag = ga$. Taking the inverse of a on the right, we obtain $g = a^{-1}ga$. Taking the inverse of a on the left we obtain $ga^{-1} = a^{-1}g$. Thus $a^{-1} \in H$ and we conclude H is closed under inverses. Thus H is a subgroup of G.

Comments: Note that g is fixed at the beginning so you cannot choose its value. The group H is called the *centralizer of* g and is denoted $Z(g)$ in the text. It is not to be confused with the *center*, which is a different subgroup!

Problem 5 Prove that if σ is a transposition in S_n then there exists a permutation $\tau \in S_n$ such that $\tau \circ \sigma \circ \tau^{-1} = (1\ 2)$.

Solution: Let $\sigma = (a\ b)$ be the transposition. Let τ be a bijective function such that $\tau(a) = 1$ and $\tau(b) = 2$. We compute that

$$(\tau \circ \sigma \circ \tau^{-1})(1) = \tau(\sigma(a)) = \tau(b) = 2$$

and

$$(\tau \circ \sigma \circ \tau^{-1})(2) = \tau(\sigma(b)) = \tau(a) = 1.$$

For $x \notin \{1, 2\}$, we have $y := \tau^{-1}(x) \notin \{a, b\}$ since τ is a bijection. Thus

$$(\tau \circ \sigma \circ \tau^{-1})(x) = \tau(\sigma(y)) = \tau(y) = x.$$

Thus $\tau \circ \sigma \circ \tau^{-1} = (1\ 2)$ as desired.

Comments: Many of you attempted to explicitly construct a formula for τ as a product of cycles. There are many ways of doing this, but you need to be careful. Taking $\sigma = (1a)(2b)$ works as long as $1 \neq a$ and $2 \neq b$ (since (11) and (22) do not make sense as cycles). Also, $\tau = (1a2b)$ works as long as $\{a, b\} \cap \{1, 2\} \neq \emptyset$; but otherwise something like (1123) does not make sense as a cycle.