You have 50 minutes to complete the exam.

Problem 1 Determine whether each of the following statements are true or false. No justification is necessary.

1. If $(G, *)$ is a group, then $*$ is associative.
2. $(\mathbb{Z}, +)$ is a group.
3. Let $(G, *)$ be a group. If $x * y = e$, then $y = x^{-1}$.
4. A group $(G, *)$ is abelian if and only if $*$ is commutative.
5. For positive integers a, b, and n, we have $a \equiv b \mod n$ if and only if a and b have the same remainder modulo n.

Problem 2 Find $\gcd(210, 45)$ and two integers x, y such that $210x + 45y = \gcd(210, 45)$.

Problem 3 Let G be the set of positive rational numbers and consider the binary operation $x * y = x^2 + y^2$ on G. Is $*$ associative? Is $*$ commutative? Does $(G, *)$ form a group?

Problem 4 List all elements of $\mathbb{Z}_{12}$ with order exactly 4.

Problem 5 Let G be a group. Prove that G is abelian if and only if $(xy)^2 = x^2y^2$ for all $x, y \in G$.