**You have 150 minutes to complete the exam.**

**Problem 1** Determine whether each of the following statements are true or false. No justification is necessary.

1. If a binary operation is associative, then it is also commutative.
2. The set $\mathbb{R}$ of real numbers form a group under multiplication.
3. If $H$ and $K$ are subgroups of a group $G$, then $H \cup K$ is a subgroup of $G$.
4. Let $G$ and $H$ be finite groups. Then $|G \times H| = |G| \times |H|$.
5. A permutation in $S_n$ is in the alternating group $A_n$ if and only if it can be written as a product of an even number of transpositions.
6. Let $H$ be a subgroup of a group $G$. The group $H$ is normal in $G$ if and only if $gHg^{-1} = H$ for every $g \in G$.
7. If $\varphi : G \to H$ is a group isomorphism, then $o(x) = o(\varphi(x))$ for every $x \in G$.
8. For every integer $n$, there is a unique isomorphism class of finite abelian group of order $n$.
9. Every zero-divisor in a ring also nilpotent.
10. Let $R$ be a commutative ring with unity. If $I$ is a maximal ideal in $R$, then $R/I$ is a field.

**Problem 2** Find the last decimal digit of $513^{234}$.

**Problem 3** Consider the binary operations $\oplus$ and $\otimes$ on $\mathbb{R}$ where $a \oplus b = \min\{a, b\}$ and $a \oplus b = x + y$. Does $(\mathbb{R}, \oplus, \otimes)$ form a ring?

**Problem 4** List all subgroups of $\mathbb{Z}_3 \times \mathbb{Z}_3$.

**Problem 5** Rewrite the permutation $\sigma = (145)(2368)(79)$ as a product of transpositions. Is $\sigma$ even or odd?

**Problem 6** Find all the zero-divisors and units in the ring $\mathbb{Z}_{12}$.

**Problem 7** Let $G$ be a group and consider the relation $\sim$ on $G$ where $x \sim y$ whenever $x$ and $y$ have the same order. Show that $\sim$ is an equivalence relation.

**Problem 8** Let $H$ and $K$ be subgroups of a finite group $G$. Prove that, if $\gcd(|H|, |K|) = 1$, then $H \cap K = \{e\}$.

**Problem 9** Let $G$ be an abelian group with subgroup $H$. Prove that $G/H$ is an abelian group.

**Problem 10** Let $R$ be a ring with unity and no nonzero zero-divisors. Prove that if $a, b \in R$ satisfy $ab = 1$, then also $ba = 1$.