You have 150 minutes to complete the exam.

Problem 1 Determine whether each of the following statements are true or false. No justification is necessary.

1. Every group G has unique identity element.
2. If G is an abelian group, then every subgroup is abelian.
3. There is exactly one subgroup of $\mathbb{Z}_n$ for each divisor of n.
4. A direct product of abelian groups is abelian.
5. A product of two odd permutations is odd.
6. Let H be a subgroup of a group G. The group H is normal in G if and only if $ghg^{-1} = h$ for every $g \in G$ and every $h \in H$.
7. If $\varphi : G \to H$ is a group homomorphism and K is a normal subgroup of G, then $\varphi(K)$ is a normal subgroup of H.
8. Every group of prime order is isomorphic to $\mathbb{Z}_n$.
9. Every nilpotent element in a ring is also a zero-divisor.
10. Every nonzero element of a field is a unit.

Problem 2 Find integers a, b such that $42x + 15y = 3$.

Problem 3 Consider the set $\mathcal{F}$ of bijective functions $f : \mathbb{R} \to \mathbb{R}$ and let $\circ$ be function composition. Does $(\mathcal{F}, \circ)$ form a group?

Problem 4 Determine the order of $(4, 6)$ in $\mathbb{Z}_{10} \times \mathbb{Z}_{15}$.

Problem 5 Rewrite the permutation $(1639)(391)(456)$ as a product of disjoint cycles.

Problem 6 Consider the subgroup $H = \{e, (12)\}$ in S_3. Determine all the distinct left cosets of H in S_3.

Problem 7 Let X be a set with a commutative binary operation. Suppose that for all $a, b, c \in G$, the equation $a * (b * c) = b * (c * a)$ holds. Prove that $*$ is associative.

Problem 8 Consider the subset

$$O_n := \left\{ A \in \mathrm{GL}(n, \mathbb{R}) \;\middle|\; A^T A = I_n \right\}.$$

Prove that O_n is a subgroup of $\mathrm{GL}(n, \mathbb{R})$.

Problem 9 Prove that the group $\mathbb{Q}/\mathbb{Z}$ has an element of every finite order.

Problem 10 Let $\varphi : \mathbb{Z}_{10} \to \mathbb{Z}_{15}$ be a function. Prove that φ is a group homomorphism if and only if there exists $c \in \mathbb{Z}$ such that $\varphi(x) = \overline{3cx}$ for all $x \in \mathbb{Z}_{10}$.