You have 150 minutes to complete the exam.

Problem 1 Determine whether each of the following statements are true or false. No justification is necessary.

1. Let G be a group and let $a, b, c \in G$. If $ab = cb$, then $a = c$.
2. The set $M_2(\mathbb{R})$ of 2×2 matrices form a group under matrix addition.
3. Every nontrivial subgroup of $\mathbb{Z}$ is infinite.
4. A direct product of cyclic groups is cyclic.
5. Every permutation in S_n can be written uniquely as a product of transpositions.
6. Let H be a subgroup of a finite group G. The number of left cosets of H in G is equal to the number of right cosets of H in G.
7. If $\varphi : G \to H$ is a group homomorphism and K is a normal subgroup of H, then $\varphi^{-1}(K)$ is a normal subgroup of G.
8. Every cyclic group is finite.
9. If R is nontrivial ring, then the zero divisors and the units are disjoint.
10. Let R be a commutative ring with unity. If I is a prime ideal in R, then R/I is a field.

Problem 2 Find integers x, y such that $34x + 15y = 1$.

Problem 3 Consider the set E of even integers with binary operation $a * b = \frac{a+b}{2}$. Is $*$ associative? Is $*$ commutative? Does $(E, *)$ form a group?

Problem 4 List all elements of $\mathbb{Z}_{24}$ of order 6.

Problem 5 Find the order of the permutation $(142)(2356)(135)$.

Problem 6 Determine all abelian groups of order 250.

Problem 7 Suppose G is a group such that $x^2 = e$ for all $x \in G$. Prove that G is abelian.

Problem 8 Let G be a group. Prove that $o(xy) = o(yx)$ for all $x, y \in G$.

Problem 9 Prove that if R is a ring with unity and $a \in R$ is a unit, then the multiplicative inverse of a is unique.

Problem 10 Consider the set of matrices

$$H := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}.$$

Prove that H is a subgroup of $\mathrm{GL}_3(\mathbb{R})$ and determine the center of H.