13.14.

By the first isomorphism theorem, $H \cong G/K$ where $K = \ker\ \varphi$. Thus $|H| = |G|/|K|$. Rearranging, we have $|H||K| = |G|$. Thus $|H|$ divides $|G|$.

14.1.

(a) Observe that $48 = 2^4 \times 3$. The partitions of 4 are 4, $3+1$, $2+2$, $2+1+1$ and $1+1+1+1$. By the fundamental theorem of finite abelian groups, the possibilities are

$$\mathbb{Z}_{16} \times \mathbb{Z}_3,\ \mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_3,\ \mathbb{Z}_4^2 \times \mathbb{Z}_3,\ \mathbb{Z}_4 \times \mathbb{Z}_2^2 \times \mathbb{Z}_3,\ \mathbb{Z}_2^4 \times \mathbb{Z}_3.$$

(b) Observe that $72 = 2^3 \times 3^2$. Thus, the possibilities are

$$\mathbb{Z}_8 \times \mathbb{Z}_9,\ \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_9,\ \mathbb{Z}_2^3 \times \mathbb{Z}_9,\ \mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3,\ \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3,\ \mathbb{Z}_2^3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$$

(c) Observe that $84 = 2^2 \times 3 \times 7$. Thus, the possibilities are

$$\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_7,\ \mathbb{Z}_2^2 \times \mathbb{Z}_3 \times \mathbb{Z}_7$$

(d) Observe that $450 = 2 \times 3^2 \times 5^2$. Thus, the possibilities are

$$\mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_{25},\ \mathbb{Z}_2 \times \mathbb{Z}_3^2 \times \mathbb{Z}_{25},\ \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5^2,\ \mathbb{Z}_2 \times \mathbb{Z}_3^2 \times \mathbb{Z}_5^2,$$

(e) Observe that $900 = 2^2 \times 3^2 \times 5^2$. Thus, the possibilities are

$$\mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_{25},\ \mathbb{Z}_4 \times \mathbb{Z}_3^2 \times \mathbb{Z}_{25},\ \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_5^2,\ \mathbb{Z}_4 \times \mathbb{Z}_3^2 \times \mathbb{Z}_5^2,$$
$$\mathbb{Z}_2^2 \times \mathbb{Z}_9 \times \mathbb{Z}_{25},\ \mathbb{Z}_2^2 \times \mathbb{Z}_3^2 \times \mathbb{Z}_{25},\ \mathbb{Z}_2^2 \times \mathbb{Z}_9 \times \mathbb{Z}_5^2,\ \mathbb{Z}_2^2 \times \mathbb{Z}_3^2 \times \mathbb{Z}_5^2,$$

14.8.

Observe that $(3,3)$ has order 3 in $G = \mathbb{Z}_9 \times \mathbb{Z}_9$. Thus $H = \langle(3,3)\rangle$ is a cyclic subgroup of G of order 3. Thus G/H has order $9^2/3 = 27$. By the Fundamental Theorem of Finite Abelian Groups, we know that G/H is isomorphic to $\mathbb{Z}_{27}$, $\mathbb{Z}_9 \times \mathbb{Z}_3$ or $\mathbb{Z}_3^3$.

Applying Theorem 12.4 to the canonical surjective homomorphism $G \to G/H$, the order of any element of G/H divides the order of some element of G. The original group has no elements of order 27, so $G/H \ncong \mathbb{Z}_{27}$.

Observe that $(1,0) \notin H$ and $3(1,0) = (3,0) \notin H$. Therefore the element $(1,0)H$ in G/H does not have order 1 or 3. Every element of $\mathbb{Z}_3^3$ has order 1 or 3, so $G/H \ncong \mathbb{Z}_3^3$.

Therefore, G/H must be isomorphic to the only remaining possiblity: $\mathbb{Z}_9 \times \mathbb{Z}_3$.

16.1.

Suppose $a \in R$. By definition of the multiplicative identity, we have $a = 1_R a$. Therefore, $a + (-1_R)a = 1_R a + (-1_R)a$. By the distributive law, this equals $(1_R + (-1_R))a$. By definition, -1_R is the additive inverse of 1_R. Thus $1_R + (-1_R) = 0_R$. By Theorem 16.1(a), $0_R a = 0_R$. Therefore $a + (-1_R)a = 0_R$. Thus $(-1_R)a$ is an additive inverse of a. Since inverses in a group are unique, $(-1_R)a = -a$.

16.2.

(a) The tuple $(\mathbb{R}, *, \Box)$ is not a ring. Observe that $2(r+s) = r$ has the unique solution $s = -r/2$ for fixed r. This solutions depends on the choice of r. Therefore, there is no element $s \in \mathbb{R}$ such that $r * s = s$ for all $r \in \mathbb{R}$. Thus $*$ does not have an additive identity. Therefore, $(\mathbb{R}, *)$ is not a group.

(b) The tuple $(\mathbb{R} - \{0\}, *, \Box)$ is not a ring. If $(\mathbb{R} - \{0\}, *)$ is a group, then the additive identity is $\frac{1}{2}$ since $r * \frac{1}{2} = 2r\frac{1}{2} = r$ for all $r \in \mathbb{R} - \{0\}$. However, $3\Box\frac{1}{2} = \frac{3}{2} \neq \frac{1}{2}$, which contradicts Theorem 16.1(a).

(c) The tuple $(\mathbb{R}^+, *, \Box)$ is not a ring. Indeed, $(2\Box 2)\Box 3 = (2^2)^3 = 64$, but $2\Box(2\Box 3) = 2^{(2^3)} = 256$. Therefore, the operation $\Box$ is not associative.

16.3.

Let R be the subset of all real numbers of the form $a + b\sqrt{2}$ for $a, b \in \mathbb{Q}$.

First, we show that R is an additive subgroup of $\mathbb{R}$. Indeed, $0 \in R$, so R is nonempty. Let $x, y \in R$. Then $x = a + b\sqrt{2}$ and $y = c + d\sqrt{2}$ for some $a, b, c, d \in \mathbb{Q}$. Now $x + y = (a + c) + (b + d)\sqrt{2}$ is in R. Therefore R is closed under addition. Finally, $-x = (-a) + (-b)\sqrt{2}$ is in R, so R is closed under additive inverses. We conclude R is an additive subgroup of $\mathbb{R}$.

Now we consider the multiplication. Let $x, y \in R$. Then $x = a + b\sqrt{2}$ and $y = c + d\sqrt{2}$ for some $a, b, c, d \in \mathbb{Q}$. We have $xy = (ac + 2bd) + (ad + bc)\sqrt{2}$ in R, so the multiplication is a well-defined binary operation on R. The distributive laws hold in R because they hold in $\mathbb{R}$. We conclude that R is a ring.

The element $1 + 0\sqrt{2}$ is the multiplicative identity in R. The ring is commutative because $\mathbb{R}$ is commutative. Finally, it remains to show that every nonzero element is a unit. Given $a + b\sqrt{2} \in R$, we observe that

$$\left(a + b\sqrt{2}\right)\left(\frac{a}{a^2 - 2b^2} - \frac{b}{a^2 - 2b^2}\sqrt{2}\right) = 1.$$

This makes sense whenever $a^2 - 2b^2 \neq 0$. Since a and b are rational numbers, this only occurs when $a = b = 0$. Thus every nonzero element is a unit. We conclude that R is a field.